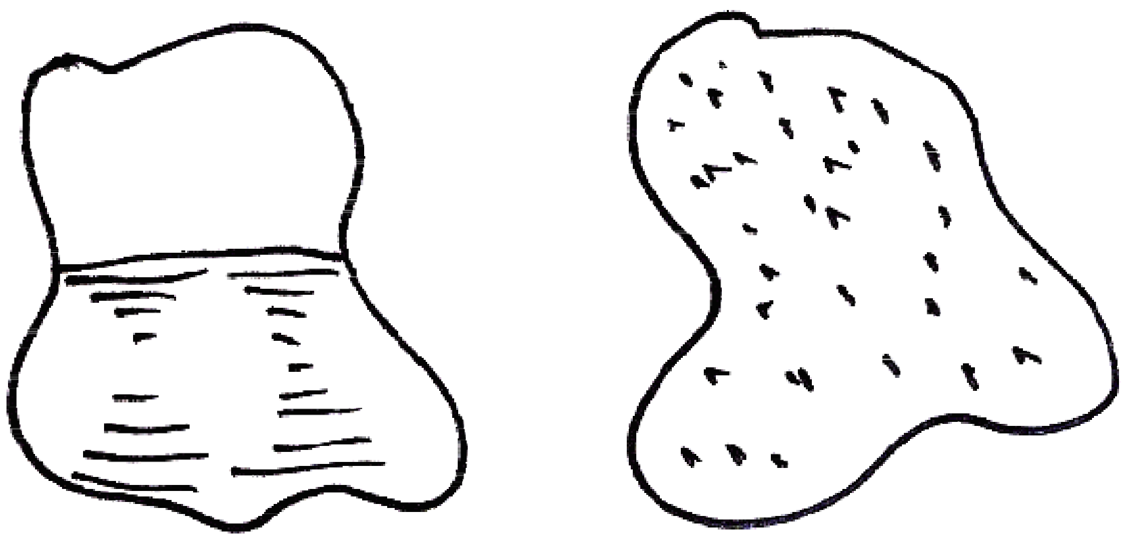
Fluid Mechanics

2nd Year

MODULE 1

Introduction to Fluids

- There are three states of matter: solids, liquids and gases.
- Both liquids and gases are classified as fluids.
- Fluids do not resist a change in shape. Therefore fluids assume the shape of the container they occupy.
- Liquids may be considered to have a fixed volume and therefore can have a free surface. Liquids are almost incompressible.
- Conversely, gases are easily compressed and will expand to fill a container they occupy.
- We will usually be interested in liquids, either at rest or in motion.



Liquid showing free surface **Gas filling volume**

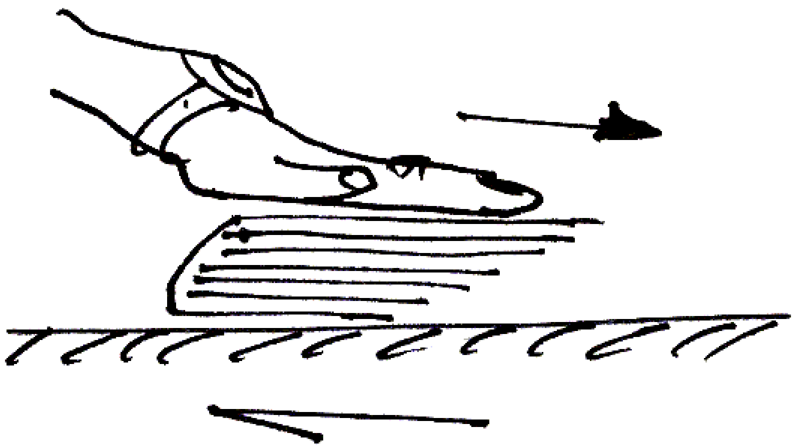
Behaviour of fluids in containers

Definition

The strict definition of a fluid is:

A fluid is a substance which conforms continuously under the action of shearing forces.

To understand this, remind ourselves of what a shear force is:



Application and effect of shear force on a book

Definition Applied to Static Fluids

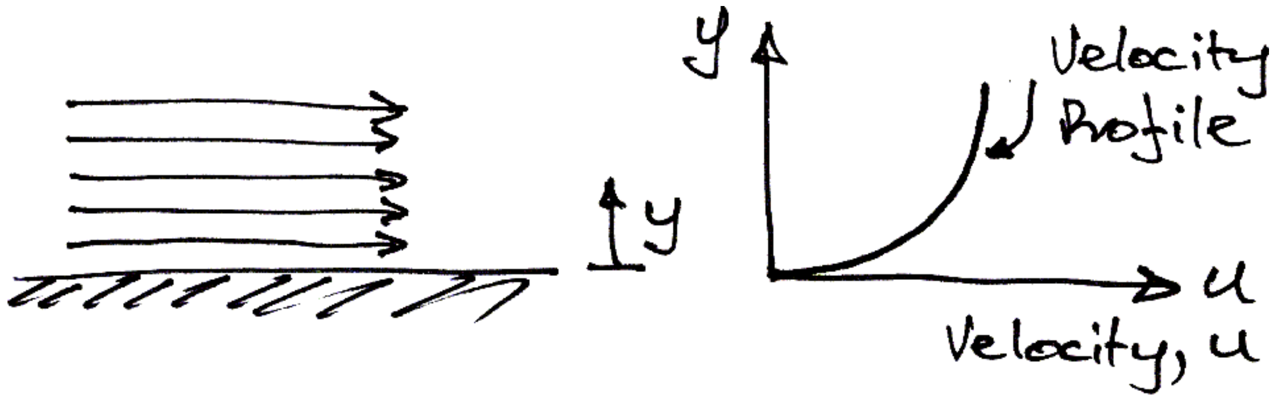
According to this definition, if we apply a shear force to a fluid it will deform and take up a state in which no shear force exists. Therefore, we can say:

If a fluid is at rest there can be no shearing forces acting and therefore all forces in the fluid must be perpendicular to the planes in which they act.

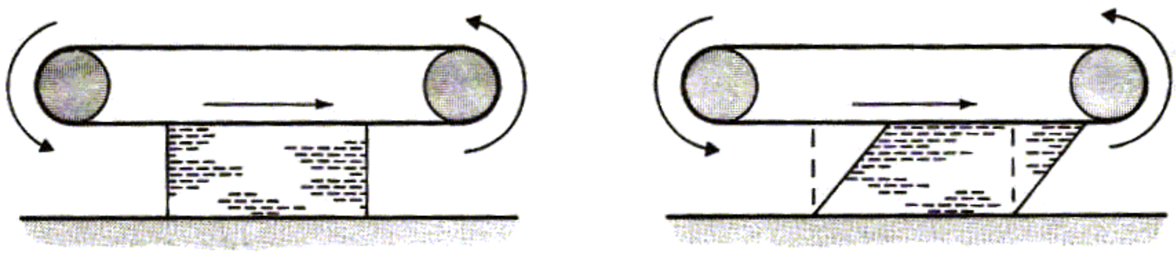
Note here that we specify that the fluid must be at rest. This is because, it is found experimentally that fluids in motion can have slight resistance to shear force. This is the source of *viscosity*.

Definition Applied to Fluids in Motion

For example, consider the fluid shown flowing along a fixed surface. At the surface there will be little movement of the fluid (it will 'stick' to the surface), whilst further away from the surface the fluid flows faster (has greater velocity):



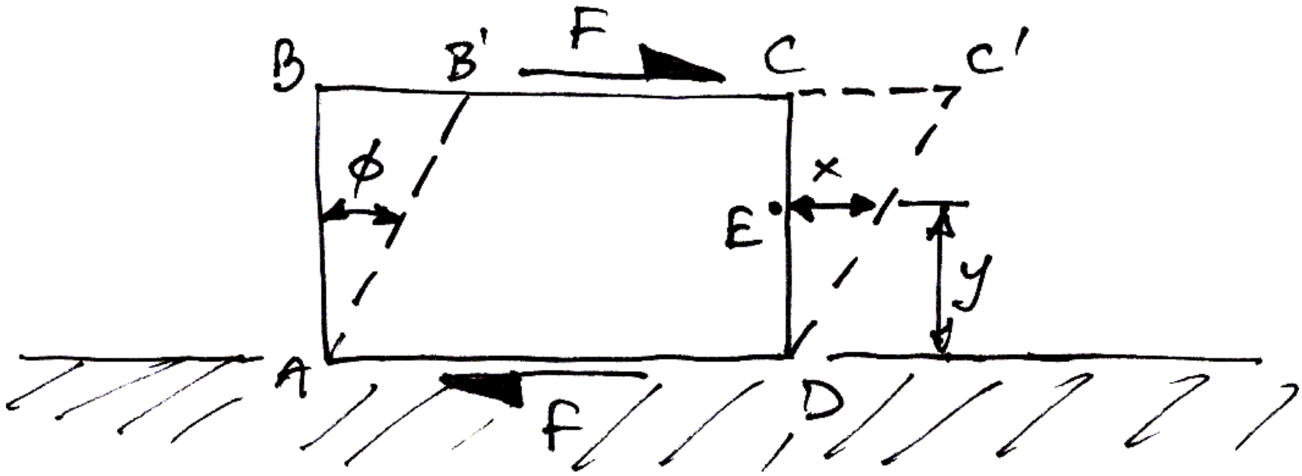
If one layer of is moving faster than another layer of fluid, there must be shear forces acting between them. For example, if we have fluid in contact with a conveyor belt that is moving we will get the behaviour shown:



Ideal fluid **Real (Viscous) Fluid**

When fluid is in motion, any difference in velocity between adjacent layers has the same effect as the conveyor belt does.

Therefore, to represent real fluids in motion we must consider the action of shear forces.



Consider the small element of fluid shown, which is subject to shear force and has a dimension s into the page. The force F acts over an area A = BC×s. Hence we have a *shear stress* applied:

$$\text{Stress} = \frac{\text{Force}}{\text{Area}}$$

$$\tau = \frac{F}{A}$$

Any stress causes a deformation, or strain, and a shear stress causes a *shear strain*. This shear strain is measured by the angle ϕ .

Remember that a fluid *continuously* deforms when under the action of shear. This is different to a solid: a solid has a single value of ϕ for each value of τ . So the longer a shear stress is applied to a fluid, the more shear strain occurs. However, what is known from experiments is that the rate of shear strain (shear strain per unit time) is related to the shear stress:

$$\text{Shear stress} \propto \text{Rate of shear strain}$$

$$\text{Shear stress} = \text{Constant} \times \text{Rate of shear strain}$$

We need to know the rate of shear strain. From the diagram, the shear strain is:

$$\phi = \frac{x}{y}$$

If we suppose that the particle of fluid at *E* moves a distance *x* in time *t*, then, using $S = R\theta$ for small angles, the rate of shear strain is:

$$\frac{\Delta\phi}{\Delta t} = \left(\frac{x}{y}\right)\Big/ t = \frac{x}{t} \cdot \frac{1}{y}$$
$$= \frac{u}{y}$$

Where *u* is the velocity of the fluid. This term is also the change in velocity with height. When we consider infinitesimally small changes in height we can write this in differential form, du/dy. Therefore we have:

$$\tau = \text{constant} \times \frac{du}{dy}$$

This constant is a property of the fluid called its dynamic viscosity (dynamic because the fluid is in motion, and viscosity because it is resisting shear stress). It is denoted μ which then gives us:

Newton's Law of Viscosity:

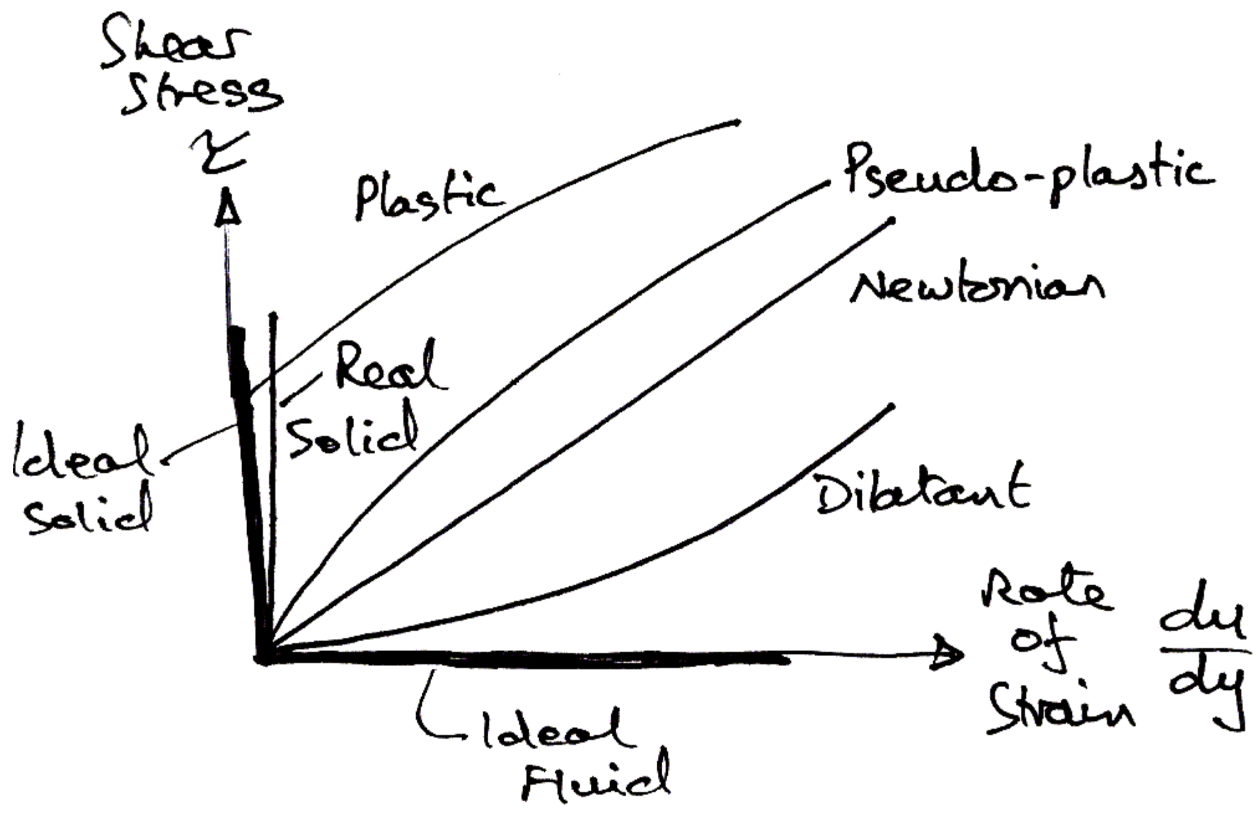
$$\boxed{\tau = \mu \frac{du}{dy}}$$

Generalized Laws of Viscosity

We have derived a law for the behaviour of fluids – that of *Newtonian fluids*. However, experiments show that there are *non-Newtonian* fluids that follow a *generalized law of viscosity*:

$$\tau = A + B\left(\frac{du}{dy}\right)^n$$

Where A, B and n are constants found experimentally. When plotted these fluids show much different behaviour to a Newtonian fluid:



Behaviour of Fluids and Solids

In this graph the Newtonian fluid is represent by a straight line, the slope of which is μ. Some of the other fluids are:

- *Plastic:* Shear stress must reach a certain minimum before flow commences.
- *Pseudo-plastic:* No minimum shear stress necessary and the viscosity decreases with rate of shear, e.g. substances like clay, milk and cement.
- *Dilatant substances;* Viscosity increases with rate of shear, e.g. quicksand.
- *Viscoelastic materials:* Similar to Newtonian but if there is a sudden large change in shear they behave like plastic.
- *Solids:* Real solids do have a slight change of shear strain with time, whereas ideal solids (those we idealise for our theories) do not.

Lastly, we also consider the *ideal fluid*. This is a fluid which is assumed to have no viscosity and is very useful for developing theoretical solutions. It helps achieve some practically useful solutions.

Units

Fluid mechanics deals with the measurement of many variables of many different types of units. Hence we need to be very careful to be consistent.

Dimensions and Base Units

The *dimension* of a measure is independent of any particular system of units. For example, velocity may be in metres per second or miles per hour, but dimensionally, it is always length per time, or $L/T = LT^{-1}$. The dimensions of the relevant base units of the Système International (SI) system are:

Unit-Free		**SI Units**	
Dimension	**Symbol**	**Unit**	**Symbol**
Mass	M	kilogram	kg
Length	L	metre	m
Time	T	second	s
Temperature	θ	kelvin	K

Derived Units

From these we have some relevant derived units (shown on the next page).

Checking the dimensions or units of an equation is very useful to minimize errors. For example, if when calculating a force and you find a pressure then you know you've made a mistake.

Quantity	Dimension	SI Unit	
		Derived	Base
Velocity	LT^{-1}	m/s	$m\,s^{-1}$
Acceleration	LT^{-2}	m/s^2	$m\,s^{-2}$
Force	MLT^{-2}	Newton, N	$kg\ m\ s^{-2}$
Pressure Stress	$ML^{-1}T^{2}$	Pascal, Pa N/m^2	$kg\ m^{-1}\ s^{-2}$
Density	ML^{-3}	kg/m^3	$kg\ m^{-3}$
Specific weight	$ML^{-2}T^{-2}$	N/m^3	$kg\ m^{-2}\ s^{-2}$
Relative density	Ratio	Ratio	Ratio
Viscosity	$ML^{-1}T^{-1}$	Ns/m^2	$kg\ m^{-1}\ s^{-1}$
Energy (work)	$ML^{2}T^{-2}$	Joule, J Nm	$kg\ m^{2}\ s^{-2}$
Power	$ML^{2}T^{-3}$	Watt, W Nm/s	$kg\ m^{2}\ s^{-3}$

Note: The acceleration due to gravity will always be taken as 9.81 m/s^2.

SI Prefixes

SI units use prefixes to reduce the number of digits required to display a quantity. The prefixes and multiples are:

Prefix Name	Prefix Unit	Multiple
Tera	T	10^{12}
Giga	G	10^{9}
Mega	M	10^{6}
Kilo	k	10^{3}
Hecto	h	10^{2}
Deka	da	10^{1}
Deci	d	10^{-1}
Centi	c	10^{-2}
Milli	m	10^{-3}
Micro	μ	10^{-6}
Nano	n	10^{-9}
Pico	p	10^{-12}

Be very particular about units and prefixes. For example:

- kN means kilo-Newton, 1000 Newtons;
- Kn is the symbol for knots – an imperial measure of speed;
- KN has no meaning;
- kn means kilo-nano – essentially meaningless.

Further Reading

- Sections 1.6 to 1.10 of *Fluid Mechanics* by Cengel & Cimbala.

Properties

Further Reading

Here we consider only the relevant properties of fluids for our purposes. Find out about surface tension and capillary action elsewhere. Note that capillary action only features in pipes of $\leq$ 10 mm diameter.

Mass Density

The mass per unit volume of a substance, usually denoted as ρ. Typical values are:

- Water: 1000 kg/m^3;
- Mercury: 13546 kg/m^3;
- Air: 1.23 kg/m^3;
- Paraffin: 800 kg/m^3.

Specific Weight

The weight of a unit volume a substance, usually denoted as γ. Essentially density times the acceleration due to gravity:

$$\gamma = \rho g$$

Relative Density (Specific Gravity)

A dimensionless measure of the density of a substance with reference to the density of some standard substance, usually water at 4°C:

$$\begin{aligned}\text{relative density} &= \frac{\text{density of substance}}{\text{density of water}} \\ &= \frac{\text{specific weight of substance}}{\text{specific weight of water}} \\ &= \frac{\rho_s}{\rho_w} = \frac{\gamma_s}{\gamma_w}\end{aligned}$$

Bulk Modulus

In analogy with solids, the bulk modulus is the modulus of elasticity for a fluid. It is the ratio of the change in unit pressure to the corresponding volume change per unit volume, expressed as:

$$\frac{\text{Change in Volume}}{\text{Original Volume}} = \frac{\text{Chnage in pressure}}{\text{Bulk Modulus}}$$

$$\frac{-dV}{V} = \frac{dp}{K}$$

Hence:

$$K = -V\frac{dp}{dV}$$

In which the negative sign indicates that the volume reduces as the pressure increases. The bulk modulus changes with the pressure and density of the fluid, but for liquids can be considered constant for normal usage. Typical values are:

- Water: 2.05 GN/m^3;
- Oil: 1.62 GN/m^3.

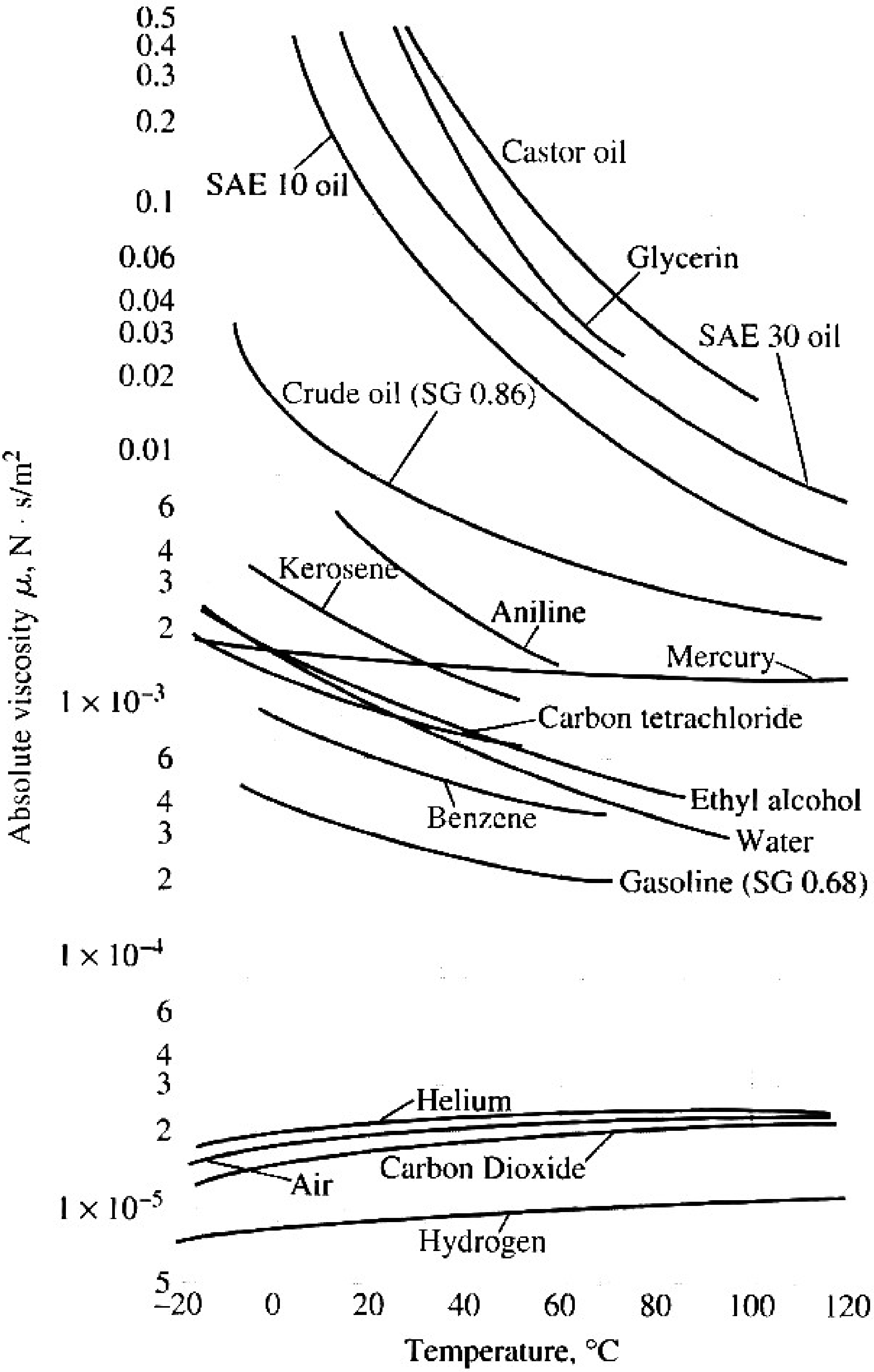
The units are the same as those of stress or pressure.

Viscosity

The viscosity of a fluid determines the amount of resistance to shear force. Viscosities of liquids decrease as temperature increases and are usually not affected by pressure changes. From Newton's Law of Viscosity:

$$\mu = \frac{\tau}{du/dy} = \frac{\text{shear stress}}{\text{rate of shear strain}}$$

Hence the units of viscosity are $\mathrm{Pa \cdot s}$ or $\mathrm{N \cdot s/m^2}$. This measure of viscosity is known as *dynamic viscosity* and some typical values are given:


0.5
0.4
0.3
0.2
0.1
0.06
0.04
0.03
0.02
0.01
6
4
3
2
1×10^{-3}
6
4
3
2
1×10^{-4}
6
4
3
2
1×10^{-5}
5
Absolute viscosity μ, N · s/m^2
SAE 10 oil
Castor oil
Glycerin
SAE 30 oil
Crude oil (SG 0.86)
Kerosene
Aniline
Mercury
Carbon tetrachloride
Ethyl alcohol
Water
Benzene
Gasoline (SG 0.68)
Helium
Carbon Dioxide
Air
Hydrogen
−20
0
20
40
60
80
100
120
Temperature, °C

Problems - Properties

a) If 6 m^3 of oil weighs 47 kN, find its specific weight, density, and relative density.

(Ans. 7.833 kN/m^3, 798 kg/m^3, 0.800)

b) At a certain depth in the ocean, the pressure is 80 MPa. Assume that the specific weight at the surface is 10 kN/m^3 and the average bulk modulus is 2.340 GPa. Find:

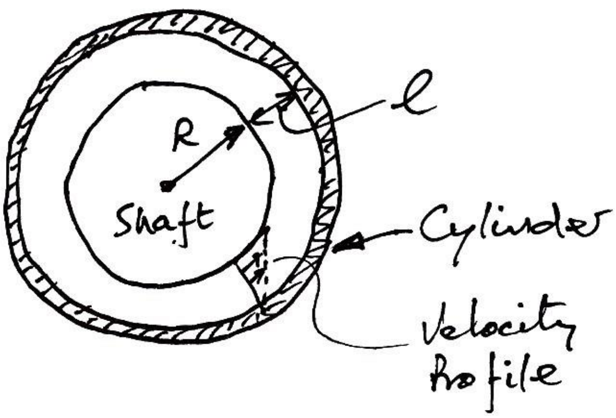
   a) the change in specific volume between the surface and the large depth;
   b) the specific volume at the depth, and;
   c) the specific weight at the depth.

(Ans. -0.335×10^{-4} m^3/kg, 9.475×10^{-4} m^3/kg, 10.35 kN/m^3)

c) A 100 mm deep stream of water is flowing over a boundary. It is considered to have zero velocity at the boundary and 1.5 m/s at the free surface. Assuming a linear velocity profile, what is the shear stress in the water?

(Ans. 0.0195 N/m^2)

d) The viscosity of a fluid is to be measured using a viscometer constructed of two 750 mm long concentric cylinders. The outer diameter of the inner cylinder is 150 mm and the gap between the two cylinders is 1.2 mm. The inner cylinder is rotated at 200 rpm and the torque is measured to be 10 Nm.



   a) Derive a generals expression for the viscosity of a fluid using this type of viscometer, and;
   b) Determine the viscosity of the fluid for the experiment above.

(Ans. 6×10^{-4} Ns/m^2)

3. Hydrostatics

Introduction

Pressure

In fluids we use the term pressure to mean:

The perpendicular force exerted by a fluid per unit area.

This is equivalent to stress in solids, but we shall keep the term pressure. Mathematically, because pressure may vary from place to place, we have:

$$p = \lim_{\Delta \to 0} \frac{\Delta F}{\Delta A}$$

As we saw, force per unit area is measured in N/m^2 which is the same as a pascal (Pa). The units used in practice vary:

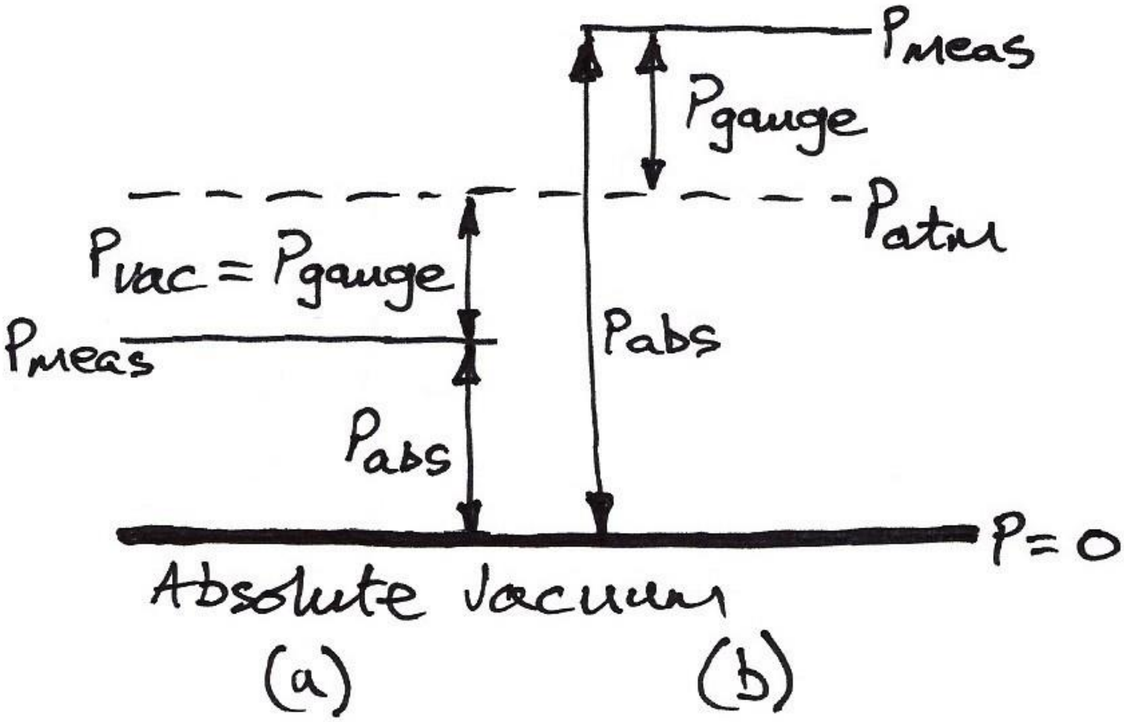
- 1 kPa = 1000 Pa = 1000 N/m^2
- 1 MPa = 1000 kPa = 1×10^6 N/m^2
- 1 bar = 10^5 Pa = 100 kPa = 0.1 MPa
- 1 atm = 101,325 Pa = 101.325 kPa = 1.01325 bars = 1013.25 millibars

For reference to pressures encountered on the street which are often imperial:

- 1 atm = 14.696 psi (i.e. pounds per square inch)
- 1 psi = 6894.7 Pa ≈ 6.89 kPa ≈ 0.007 MPa

Pressure Reference Levels

The pressure that exists anywhere in the universe is called the *absolute pressure*, P_{abs}. This then is the amount of pressure greater than a pure vacuum. The atmosphere on earth exerts *atmospheric pressure*, P_{atm}, on everything in it. Often when measuring pressures we will calibrate the instrument to read zero in the open air. Any measured pressure, P_{meas}, is then a positive or negative deviation from atmospheric pressure. We call such deviations a *gauge pressure*, P_{gauge}. Sometimes when a gauge pressure is negative it is termed a *vacuum pressure*, P_{vac}.



The above diagram shows:

(a) the case when the measured pressure is below atmospheric pressure and so is a negative gauge pressure or a vacuum pressure;

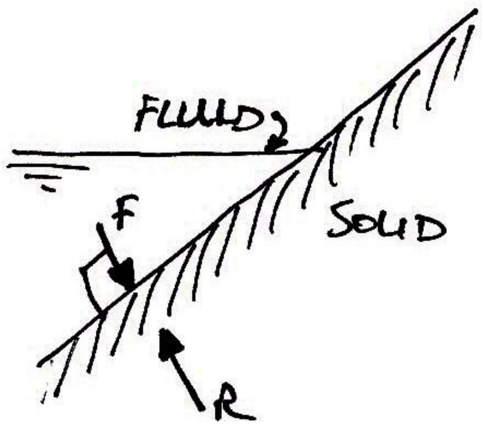
(b) the more usual case when the measured pressure is greater than atmospheric pressure by the gauge pressure.

Pressure in a Fluid

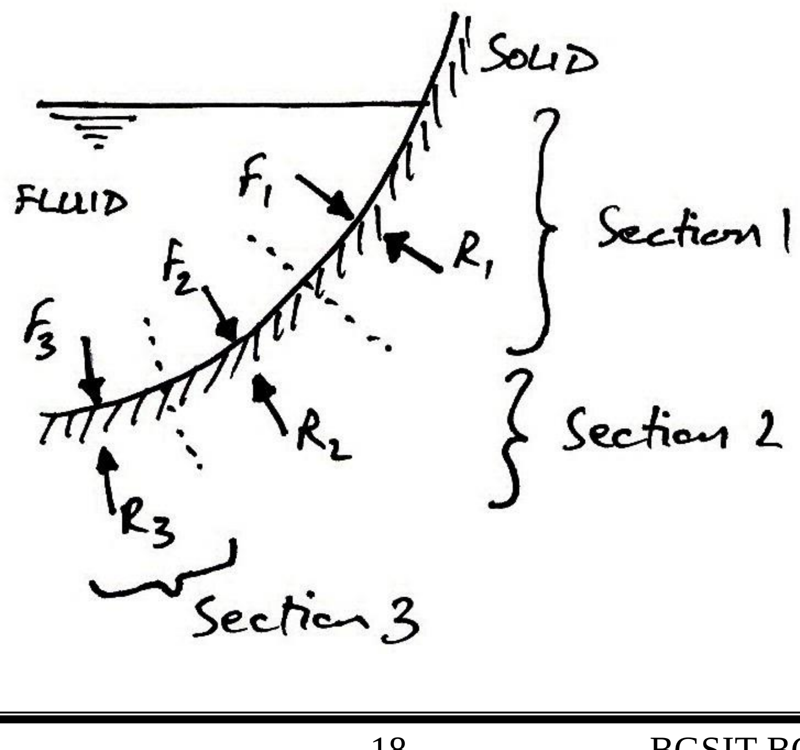
Statics of Definition

We applied the definition of a fluid to the static case previously and determined that there must be no shear forces acting and thus only forces normal to a surface act in a fluid.

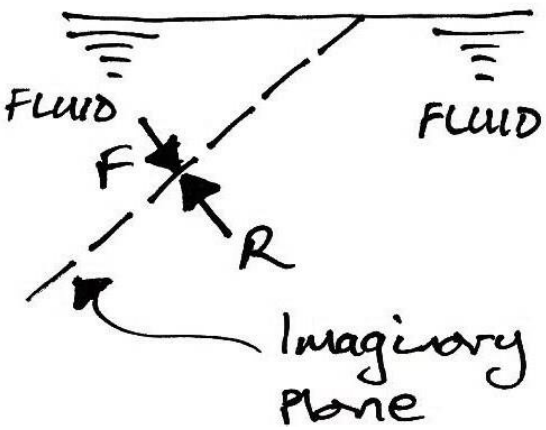
For a flat surface at arbitrary angle we have:



A curved surface can be examined in sections:



And we are not restricted to actual solid-fluid interfaces. We can consider imaginary planes through a fluid:

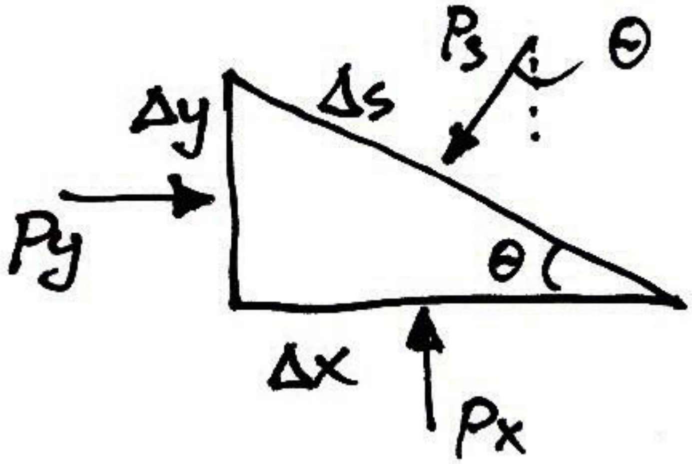


Pascal's Law

This law states:

The pressure at a point in a fluid at rest is the same in all directions.

To show this, we will consider a very small wedge of fluid surrounding the point. This wedge is unit thickness into the page:



As with all static objects the forces in the *x* and *y* directions should balance. Hence:

$$\sum F_x = 0: \qquad p_y \cdot \Delta y - p_s \cdot \Delta s \cdot \sin\theta = 0$$

But $\sin\theta = \dfrac{\Delta y}{\Delta s}$, therefore:

$$p_y \cdot \Delta y - p_s \cdot \Delta s \cdot \frac{\Delta y}{\Delta s} = 0$$

$$p_y \cdot \Delta y = p_s \cdot \Delta y$$

$$p_y = p_s$$

$$\sum F_y = 0: \qquad p_x \cdot \Delta x - p_s \cdot \Delta s \cdot \cos\theta = 0$$

But $\cos\theta = \dfrac{\Delta x}{\Delta s}$, therefore:

$$p_x \cdot \Delta x - p_s \cdot \Delta s \cdot \frac{\Delta x}{\Delta s} = 0$$

$$p_x \cdot \Delta x = p_s \cdot \Delta x$$

$$p_x = p_s$$

Hence for any angle:

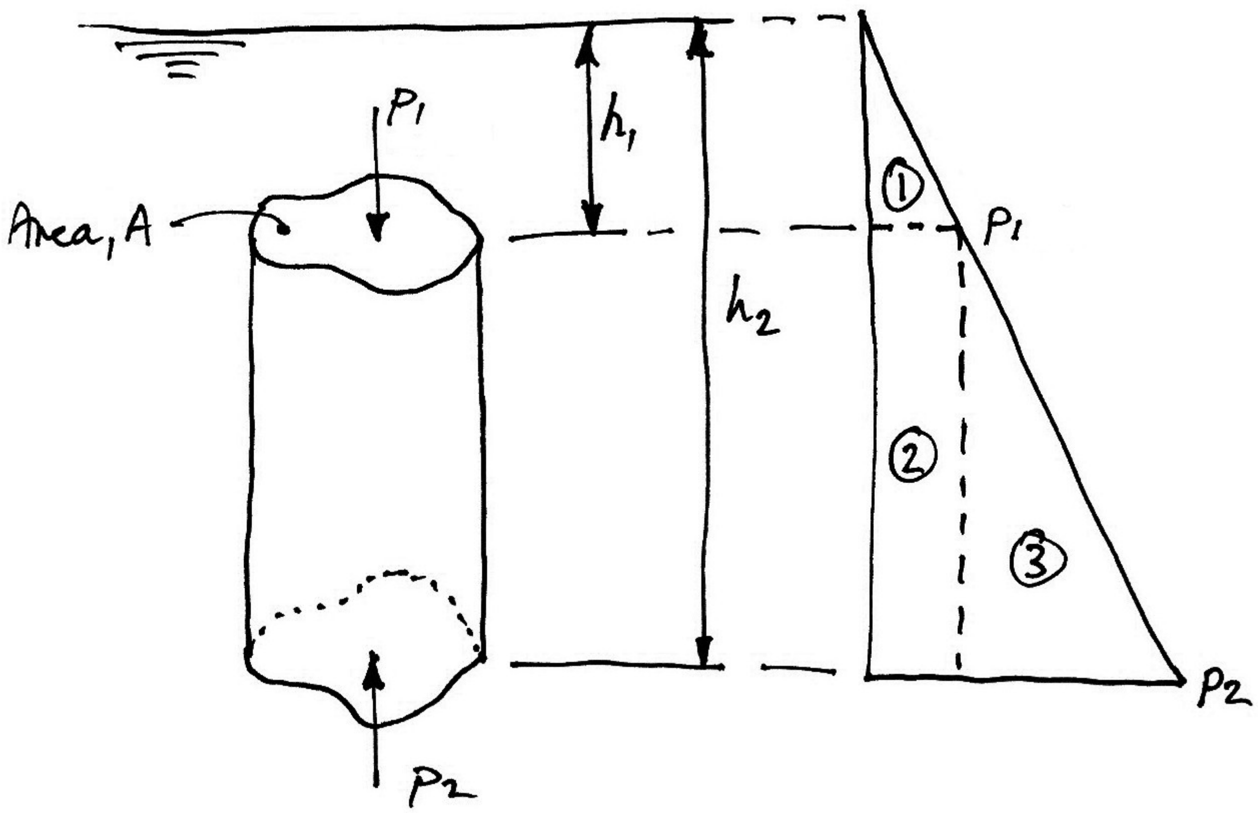
$$p_y = p_x = p_s$$

And so the pressure at a point is the same in any direction. Note that we neglected the weight of the small wedge of fluid because it is infinitesimally small. This is why Pascal’s Law is restricted to the pressure at a point.

Pressure Variation with Depth

Pressure in a static fluid does not change in the horizontal direction as the horizontal forces balance each other out. However, pressure in a static fluid does change with depth, due to the extra weight of fluid on top of a layer as we move downwards.

Consider a column of fluid of arbitrary cross section of area, *A*:



Column of Fluid **Pressure Diagram**

Considering the weight of the column of water, we have:

$$\sum F_y = 0: \qquad p_1 A + \gamma A(h_2 - h_1) - p_2 A = 0$$

Obviously the area of the column cancels out: we can just consider pressures. If we say the height of the column is $h = h_2 - h_1$ and substitute in for the specific weight, we see the difference in pressure from the bottom to the top of the column is:

$$p_2 - p_1 = \rho g h$$

This difference in pressure varies linearly in *h*, as shown by the Area 3 of the pressure diagram. If we let $h_1 = 0$ and consider a gauge pressure, then $p_1 = 0$ and we have:

$$p_2 = \rho g h$$

Where *h* remains the height of the column. For the fluid on top of the column, this is the source of p_1 and is shown as Area 1 of the pressure diagram. Area 2 of the pressure diagram is this same pressure carried downwards, to which is added more pressure due to the extra fluid.

To summarize:

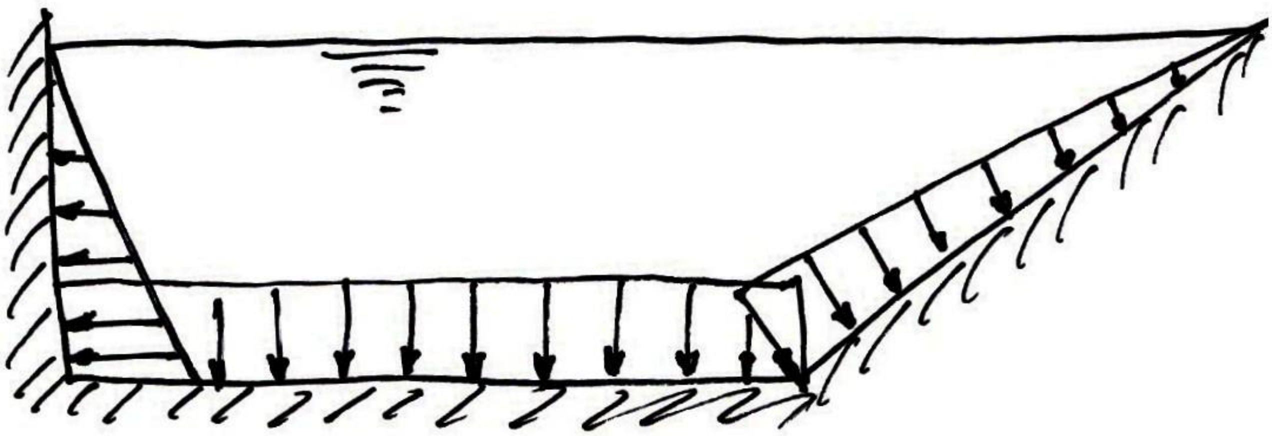
The gauge pressure at any depth from the surface of a fluid is:

$$\boxed{p = \rho g h}$$

Summary

1. Pressure acts normal to any surface in a static fluid;
2. Pressure is the same at a point in a fluid and acts in all directions;
3. Pressure varies linearly with depth in a fluid.

By applying these rules to a simple swimming pool, the pressure distribution around the edges is as shown:

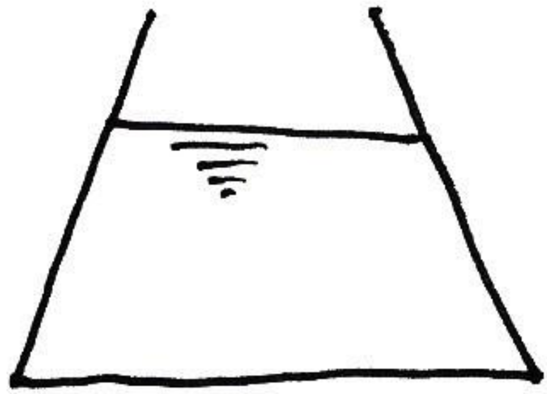


Note:

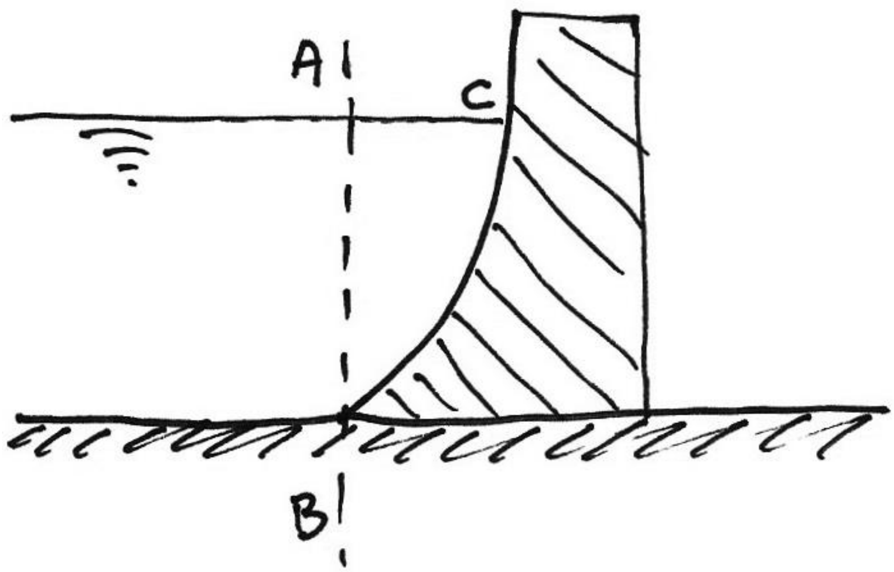
1. Along the bottom the pressure is constant due to a constant depth;
2. Along the vertical wall the pressure varies linearly with depth and acts in the horizontal direction;
3. Along the sloped wall the pressure again varies linearly with depth but also acts normal to the surface;
4. At the junctions of the walls and the bottom the pressure is the same.

Problems - Pressure

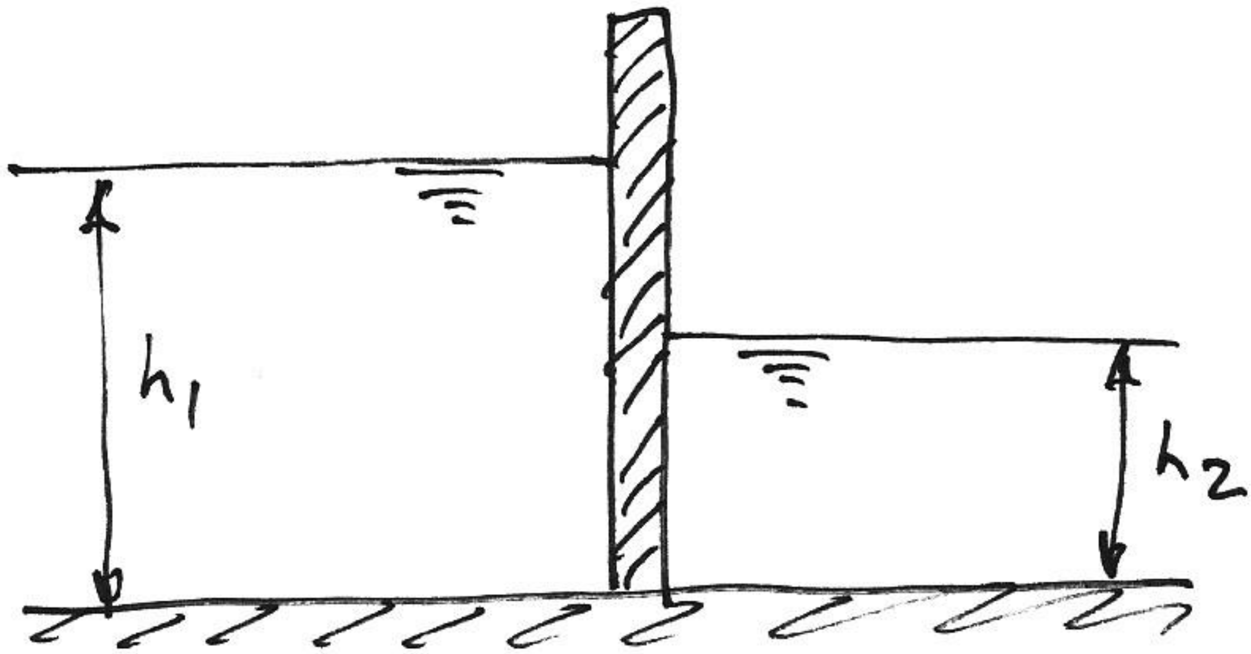
1. Sketch the pressure distribution applied to the container by the fluid:



2. For the dam shown, sketch the pressure distribution on line *AB* and on the surface of the dam, *BC*. Sketch the resultant force on the dam.



3. For the canal gate shown, sketch the pressure distributions applied to it. Sketch the resultant force on the gate? If $h_1 = 6.0$ m and $h_2 = 4.0$ m, sketch the pressure distribution to the gate. Also, what is the value of the resultant force on the gate and at what height above the bottom of the gate is it applied?



Pressure Measurement

Pressure Head

Pressure in fluids may arise from many sources, for example pumps, gravity, momentum etc. Since $p = \rho gh$, a height of liquid column can be associated with the pressure p arising from such sources. This height, h, is known as the pressure head.

Example:

The gauge pressure in a water mains is 50 kN/m^2, what is the pressure head?

The pressure head equivalent to the pressure in the pipe is just:

$$
\begin{aligned}
p &= \rho g h \\
h &= \frac{p}{\rho g} \\
&= \frac{50 \times 10^3}{1000 \times 9.81} \\
&\approx 5.1 \text{ m}
\end{aligned}
$$

So the pressure at the bottom of a 5.1 m deep swimming pool is the same as the pressure in this pipe.

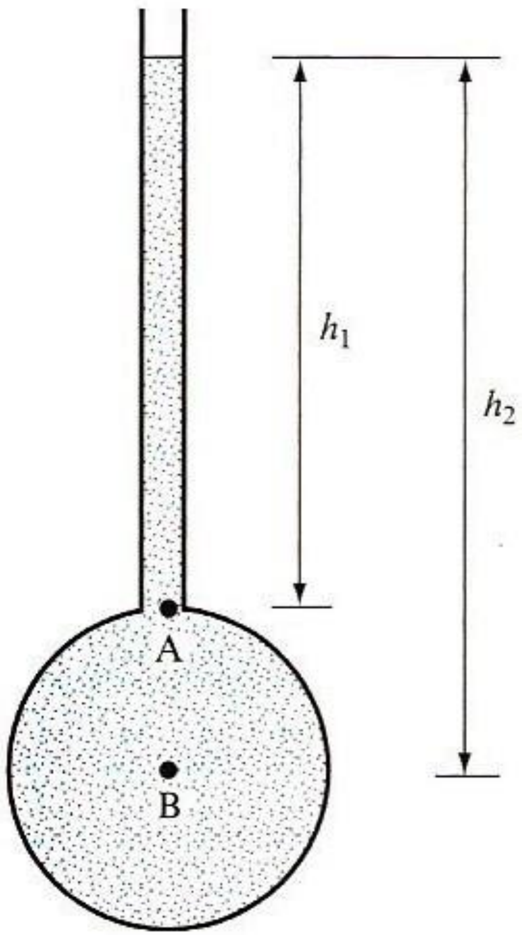
Manometers

A manometer (or liquid gauge) is a pressure measurement device which uses the relationship between pressure and head to give readings.

In the following, we wish to measure the pressure of a fluid in a pipe.

Piezometer

This is the simplest gauge. A small vertical tube is connected to the pipe and its top is left open to the atmosphere, as shown.



The pressure at *A* is equal to the pressure due to the column of liquid of height h_1:

$$p_A = \rho g h_1$$

Similarly,

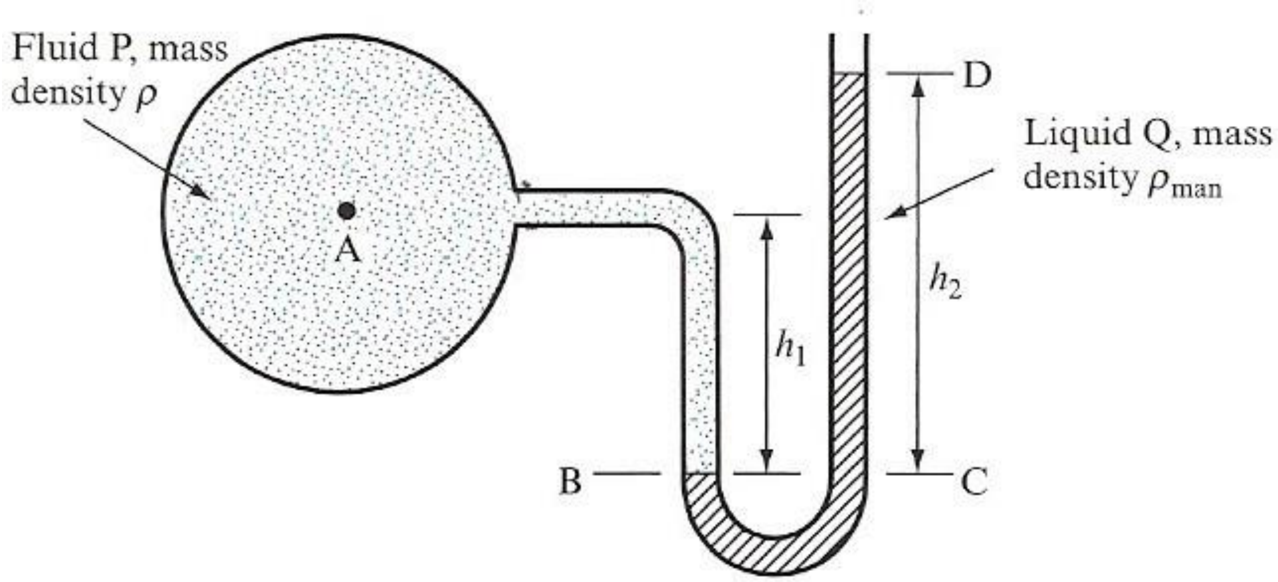
$$p_B = \rho g h_2$$

The problem with this type of gauge is that for usual civil engineering applications the pressure is large (e.g. 100 kN/m^2) and so the height of the column is impractical (e.g.10 m).

Also, obviously, such a gauge is useless for measuring gas pressures.

U-tube Manometer

To overcome the problems with the piezometer, the U-tube manometer seals the fluid by using a measuring (manometric) liquid:



Choosing the line *BC* as the interface between the measuring liquid and the fluid, we know:

$$\text{Pressure at } B,\ p_B = \text{Pressure at } C,\ p_C$$

For the left-hand side of the U-tube:

$$p_B = p_A + \rho g h_1$$

For the right hand side:

$$p_C = \rho_{man} g h_2$$

Where we have ignored atmospheric pressure and are thus dealing with gauge pressures. Thus:

$$p_B = p_C$$
$$p_A + \rho g h_1 = \rho_{man} g h_2$$

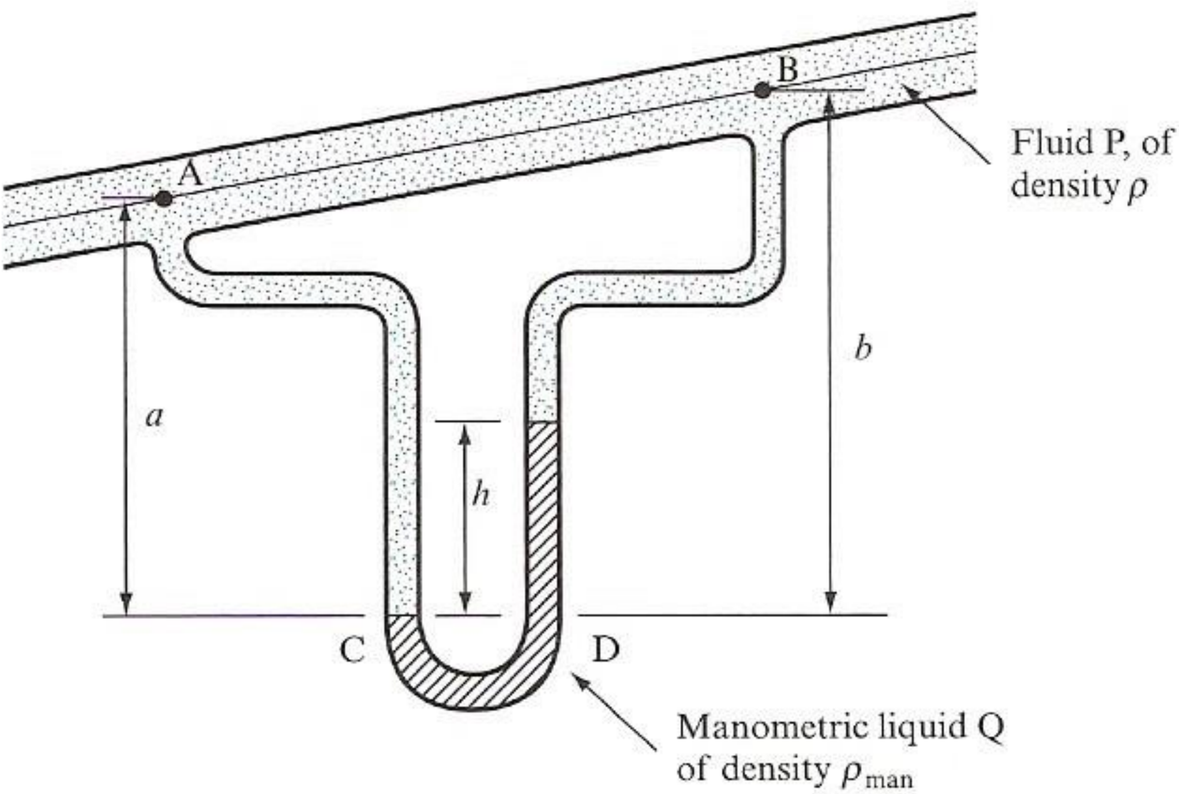
And so:

$$p_A = \rho_{man} g h_2 - \rho g h_1$$

Notice that we have used the fact that in any continuous fluid, the pressure is the same at any horizontal level.

Differential Manometer

To measure the pressure difference between two points we use a u-tube as shown:



Using the same approach as before:

$$\text{Pressure at } C,\ p_C = \text{Pressure at } D,\ p_D$$

$$p_A + \rho g a = p_B + \rho g (b - h) + \rho_{man} g h$$

Hence the pressure difference is:

$$p_A - p_B = \rho g (b - a) + hg (\rho_{man} - \rho)$$

Problems – Pressure Measurement

1. What is the pressure head, in metres of water, exerted by the atmosphere?

(Ans. 10.3 m)

2. What is the maximum gauge pressure of water that can be measured using a piezometer 2.5 m high?

(Ans. 24.5 kN/m^2)

3. A U-tube manometer is used to measure the pressure of a fluid of density 800 kg/m^3. If the density of the manometric liquid is 13.6×10^3 kg/m^3, what is the gauge pressure in the pipe if

   (a) $h_1 = 0.5$ m and D is 0.9 m above BC;

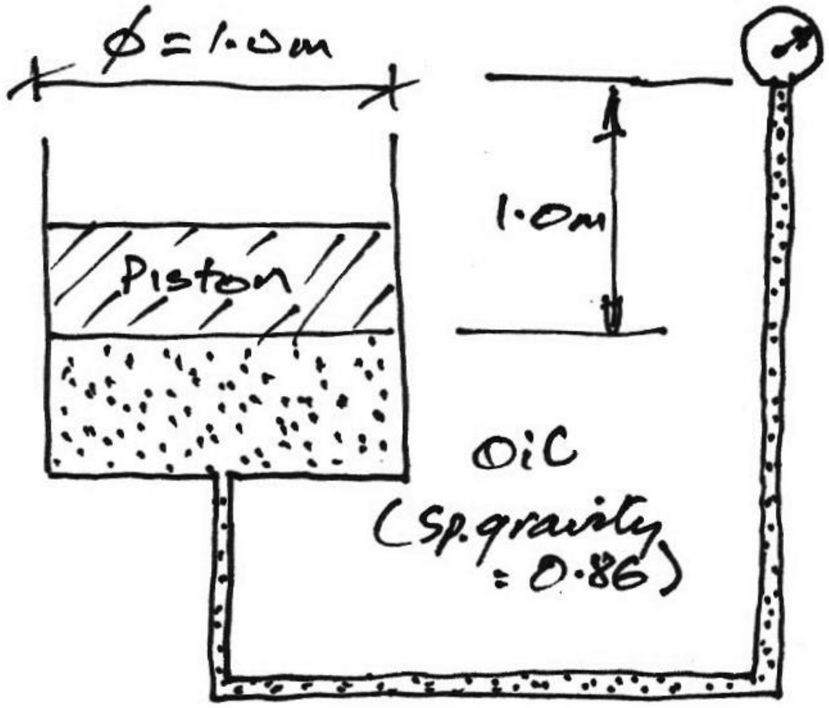
   (b) $h_1 = 0.1$ m and D is 0.2 m below BC?

(Ans. 116.15 kN/m^2, -27.45 kN/m^2)

4. A differential manometer is used to measure the pressure difference between two points in a pipe carrying water. The manometric liquid is mercury and the points have a 0.3 m height difference. Calculate the pressure difference when $h = 0.7$ m.

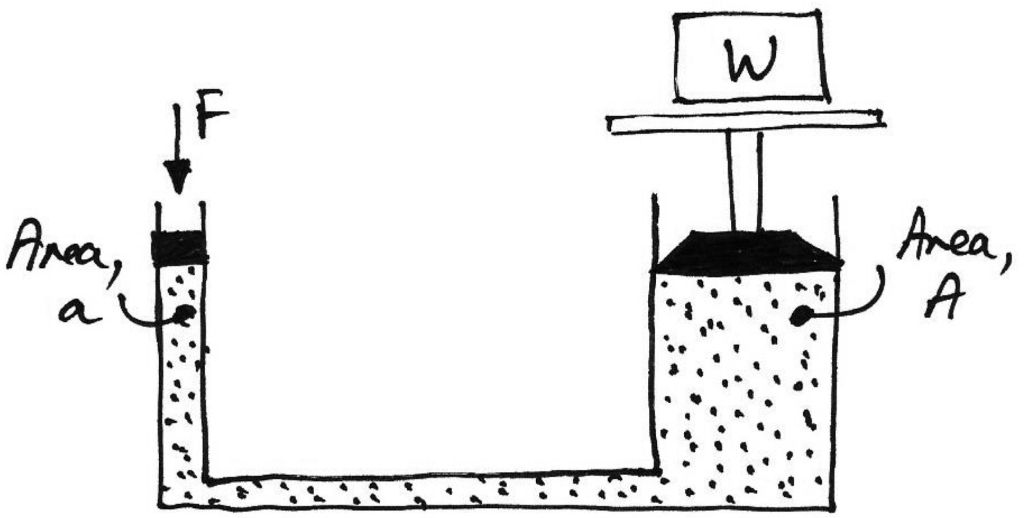
(Ans. 89.47 kN/m^2)

5. For the configuration shown, calculate the weight of the piston if the gauge pressure reading is 70 kPa.



(Ans. 61.6 kN)

6. A hydraulic jack having a ram 150 mm in diameter lifts a weight W = 20 kN under the action of a 30 mm plunger. What force is required on the plunger to lift the weight?

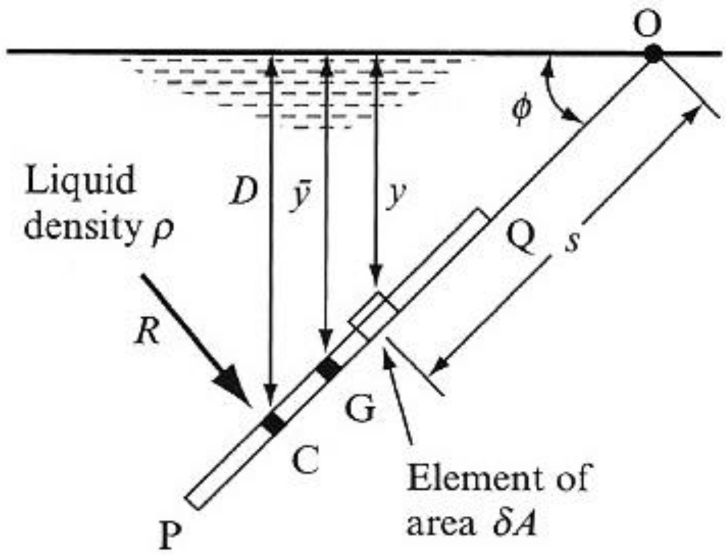


(Ans. 800 N)

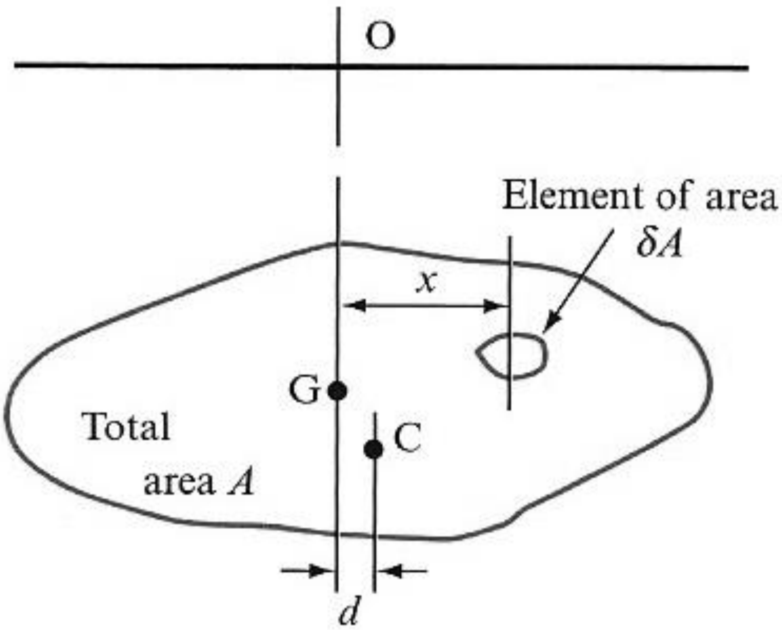
Fluid Action on Surfaces

Plane Surfaces

We consider a plane surface, *PQ*, of area *A*, totally immersed in a liquid of density ρ and inclined at an angle ϕ to the free surface:



Side Elevation



Front Elevation

If the plane area is symmetrical about the vertical axis *OG*, then $d = 0$. We will assume that this is normally the case.

Find Resultant Force:

The force acting on the small element of area, δA, is:

$$\delta R = p \cdot \delta A = \rho g y \cdot \delta A$$

The total force acting on the surface is the sum of all such small forces. We can integrate to get the force on the entire area, but remember that *y* is not constant:

$$\begin{aligned} R &= \int \rho g y \cdot \delta A \\ &= \rho g \int y \cdot \delta A \end{aligned}$$

But $\int y \cdot \delta A$ is just the first moment of area about the surface. Hence:

$$R = \rho g A \bar{y}$$

Where $\bar{y}$ is the distance to the centroid of the area (point *G*) from the surface.

Vertical Point Where Resultant Acts:

The resultant force acts perpendicular to the plane and so makes an angle $90° - \phi$ to the horizontal. It also acts through point *C*, the centre of pressure, a distance *D* below the free surface. To determine the location of this point we know:

$$\text{Moment of } R \text{ about } O \quad = \quad \begin{array}{l} \text{Sum of moments of forces} \\ \text{on all elements about } O \end{array}$$

Examining a small element first, and since $y = s\sin\phi$, the moment is:

$$\text{Moment of } \delta R \text{ about } O = \left[\rho g\left(s\sin\phi\right)\cdot\delta A\right]s$$
$$= \rho g \sin\phi\left(s^2\cdot\delta A\right)$$

In which the constants are taken outside the bracket. The total moment is thus:

$$\text{Moment of } R \text{ about } O = \rho g \sin\phi\cdot\int s^2\cdot\delta A$$

But $\int s^2\cdot\delta A$ is the second moment of area about point O or just I_O. Hence we have:

$$\text{Moment of } R \text{ about } O = \rho g \sin\phi\cdot I_O$$
$$\rho gA\bar{y}\times OC = \rho g\sin\phi\cdot I_O$$
$$A\bar{y}\times\frac{D}{\sin\phi} = \sin\phi\cdot I_O$$
$$D = \frac{I_O}{A\bar{y}}\cdot\sin^2\phi$$

If we introduce the parallel axis theorem:

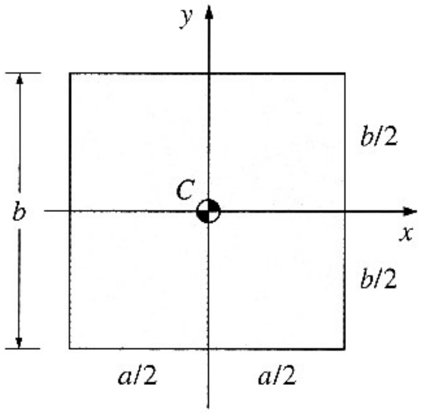
$$I_O = I_G + A\times\left(OG\right)^2$$
$$= I_G + A\cdot\left(\frac{\bar{y}}{\sin\phi}\right)^2$$

Hence we have:

$$D = \frac{I_G + A\bar{y}^2}{A\bar{y}}\cdot\frac{\sin^2\phi}{\sin^2\phi}$$
$$= \bar{y} + \frac{I_G}{A\bar{y}}$$

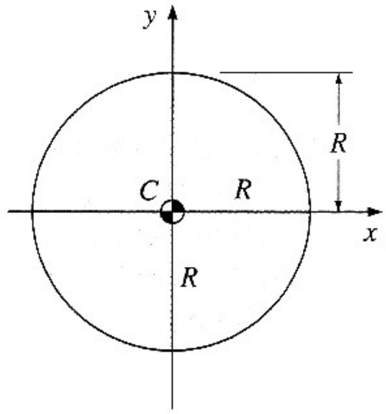
Hence, the centre of pressure, point *C*, always lies below the centroid of the area, *G*.

Plane Surface Properties



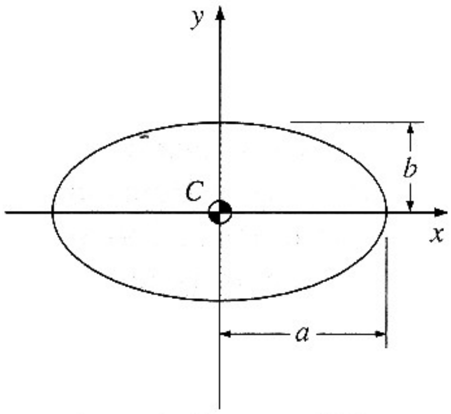
$A = ab,\ I_{xx,\,C} = ab^3/12$

(*a*) Rectangle



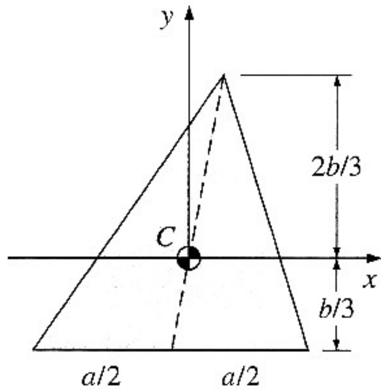
$A = \pi R^2,\ I_{xx,\,C} = \pi R^4/4$

(*b*) Circle



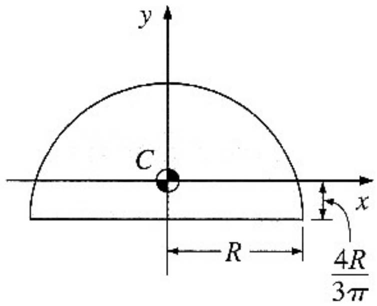
$A = \pi ab,\ I_{xx,\,C} = \pi ab^3/4$

(*c*) Ellipse



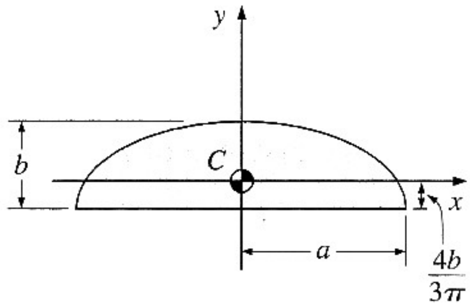
$A = ab/2,\ I_{xx,\,C} = ab^3/36$

(*d*) Triangle



$A = \pi R^2/2,\ I_{xx,\,C} = 0.109757R^4$

(*e*) Semicircle



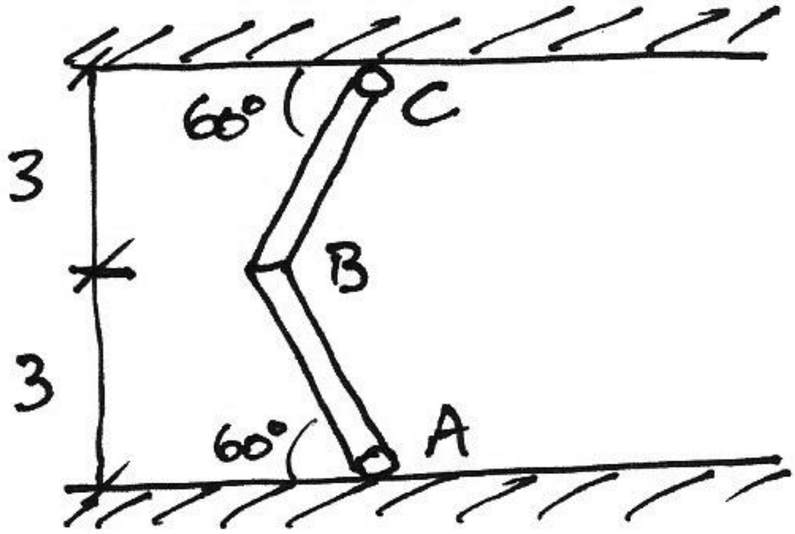
$A = \pi ab/2,\ I_{xx,\,C} = 0.109757ab^3$

(*f*) Semiellipse

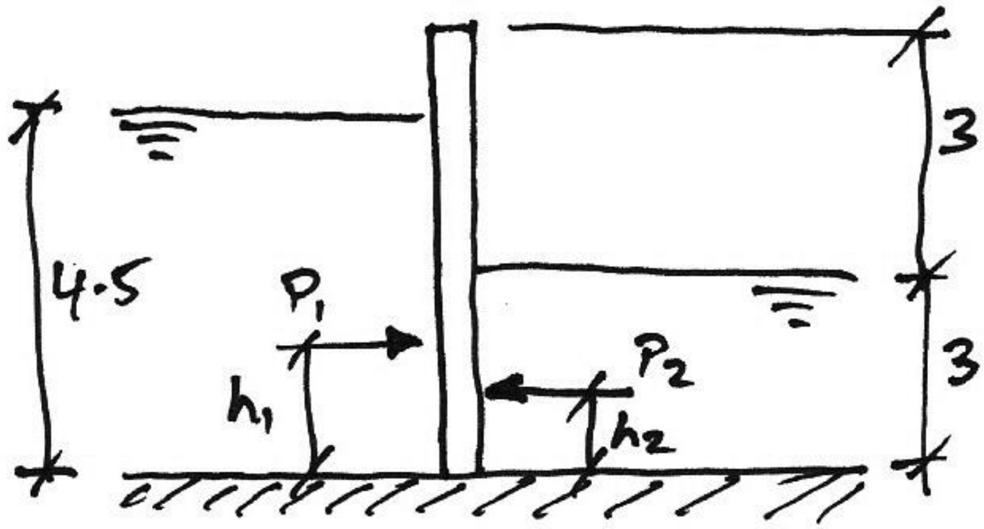
Plane Surfaces – Example

Problem

Calculate the forces on the hinges supporting the canal gates as shown. The hinges are located 0.6 m from the top and bottom of each gate.



Plan



Elevation

Solution

We will consider gate *AB*, but all arguments will equally apply to gate *BC*.

The length of the gate is $L = 3.0/\sin 30^\circ = 3.464\ \text{m}$. The resultant pressure on the gate from the high water side is:

$$\begin{aligned} P_1 &= \rho g A_1 \bar{y}_1 \\ &= 10^3 \times 9.81 \times (3.464 \times 4.5) \times \frac{4.5}{2} \\ &= 344\ \text{kN} \end{aligned}$$

Similarly for the low water side:

$$\begin{aligned} P_2 &= \rho g A_2 \bar{y}_2 \\ &= 10^3 \times 9.81 \times (3.464 \times 3.0) \times \frac{3.0}{2} \\ &= 153\ \text{kN} \end{aligned}$$

The net resultant force on the gate is:

$$P = P_1 - P_2 = 344 - 153 = 191\ \text{kN}$$

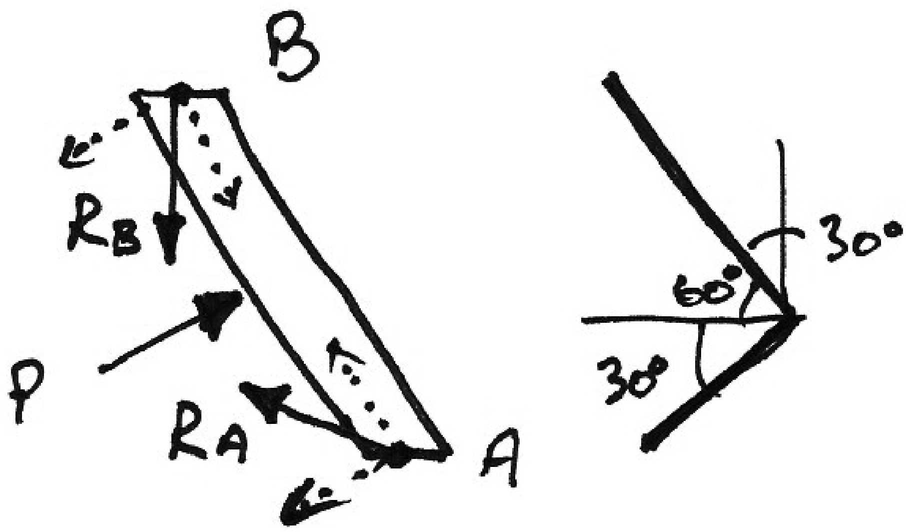
To find the height at which this acts, take moments about the bottom of the gate:

$$\begin{aligned} Ph &= P_1 h_1 + P_2 h_2 \\ &= 344 \times \frac{4.5}{3} - 153 \times \frac{3}{3} = 363\ \text{kNm} \end{aligned}$$

Hence:

$$h = \frac{363}{191} = 1.900\ \text{m}$$

Examining a free-body diagram of the gate, we see that the interaction force between the gates, R_B, is shown along with the total hinge reactions, R_A and the net applied hydrostatic force, *P*. Relevant angles are also shown. We make one assumption: the interaction force between the gates acts perpendicular on the contact surface between the gates. Hence R_B acts vertically downwards on plan.



From statics we have $\sum \text{Moments about } A = 0$:

$$P \cdot \frac{L}{2} + \left(R_B \sin 30^\circ\right) L = 0$$

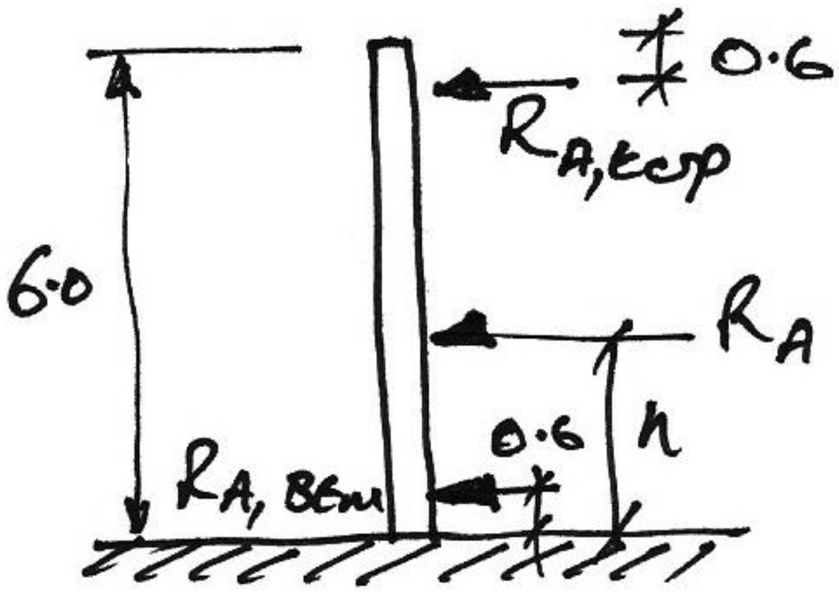
$$R_B \cdot \frac{1}{2} = \frac{P}{2}$$

$$R_B = P$$

Hence $R_B = 191\ \text{kN}$ and the component of R_B perpendicular to the gate is 95.5 kN.

By the sum of forces perpendicular to the gate, the component of R_A perpendicular to the gate must also equal 95.5 kN. Further, taking the sum of forces along the gate, the components of both R_A and R_B must balance and so $R_A = R_B = 191\ \text{kN}$.

The resultant forces R_A and R_B must act at the same height as P in order to have static equilibrium. To find the force on each hinge at A, consider the following figure:



Taking moments about the bottom hinge:

$$R_A\left(h-0.6\right)-R_{A,top}\left(6-0.6-0.6\right)=0$$
$$R_{A,top}=\frac{191\left(1.900-0.6\right)}{4.8}=51.7\ \text{kN}$$

And summing the horizontal forces:

$$R_A=R_{A,top}+R_{A,btm}$$
$$R_{A,btm}=191-51.7=139.3\ \text{kN}$$

It makes intuitive sense that the lower hinge has a larger force. To design the bolts connecting the hinge to the lock wall the direct tension and shear forces are required. Calculate these for the lower hinge.

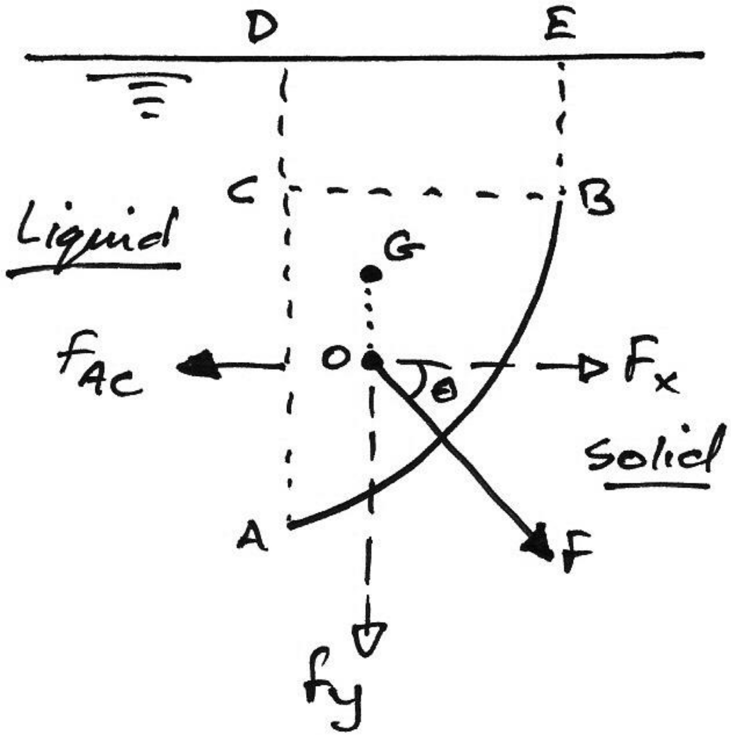
(Ans. T = 120.6 kN, V = 69.7 kN)

Curved Surfaces

For curved surfaces the fluid pressure on the infinitesimal areas are not parallel and so must be combined vectorially. It is usual to consider the total horizontal and vertical force components of the resultant.

Surface Containing Liquid

Consider the surface AB which contains liquid as shown below:



- Horizontal Component

Using the imaginary plane *ACD* we can immediately see that the horizontal component of force on the surface must balance with the horizontal force F_{AC}. Hence:

$$F_x = \text{Force on projection of surface onto a vertical plane}$$

F_x must also act at the same level as F_{AC} and so it acts through the centre of pressure of the projected surface.

- Vertical Component

The vertical component of force on the surface must balance the weight of liquid above the surface. Hence:

$$F_y = \text{Weight of liquid directly above the surface}$$

Also, this component must act through the centre of gravity of the area *ABED*, shown as *G* on the diagram.

- Resultant

The resultant force is thus:

$$F = \sqrt{F_x^2 + F_y^2}$$

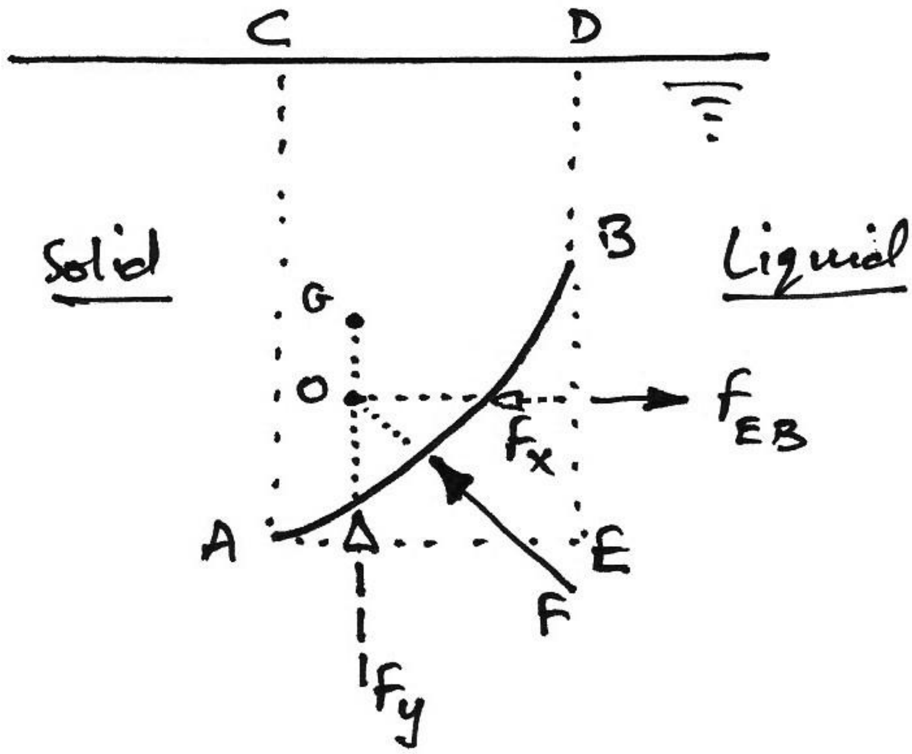
This force acts through the point *O* when the surface is uniform into the page, at an angle of:

$$\theta = \tan^{-1} \frac{F_y}{F_x}$$

to the horizontal. Depending on whether the surface contains or displaces water the angle is measured clockwise (contains) or anticlockwise (displaces) from the horizontal.

Surface Displacing Liquid

Consider the surface AB which displaces liquid as shown below:



- Horizontal Component

Similarly to the previous case, the horizontal component of force on the surface must balance with the horizontal force F_{EB}. Hence again:

$$F_x = \begin{array}{l}\text{Force on projection of surface}\\ \text{onto a vertical plane}\end{array}$$

This force also acts at the same level as F_{EB} as before.

- Vertical Component

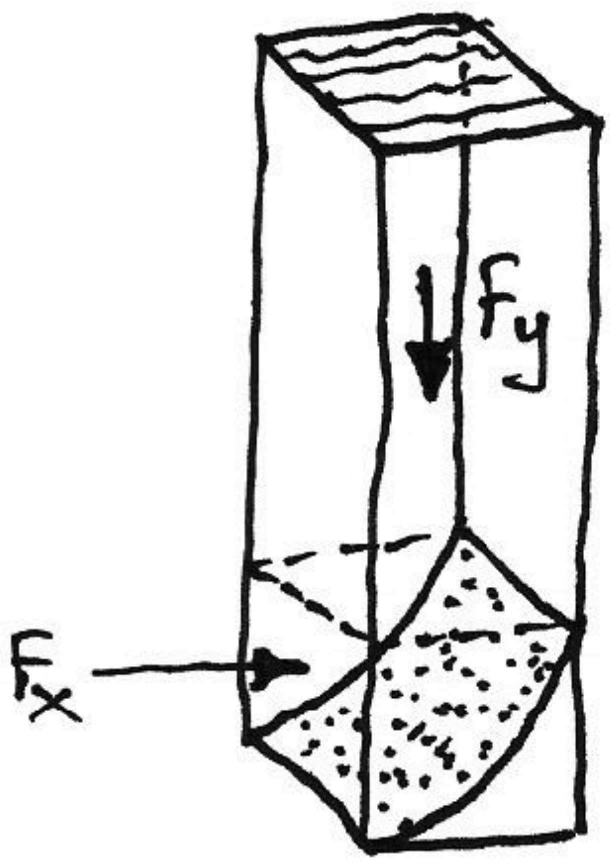
In this case we imagine that the area *ABDC* is filled with the same liquid. In this case F_y would balance the weight of the liquid in area *ABDC*. Hence:

$$F_y = \text{Weight of liquid which would lie above the surface}$$

This component acts through the centre of gravity of the imaginary liquid in area *ABDC*, shown as *G* on the diagram.

The resultant force is calculated as before.

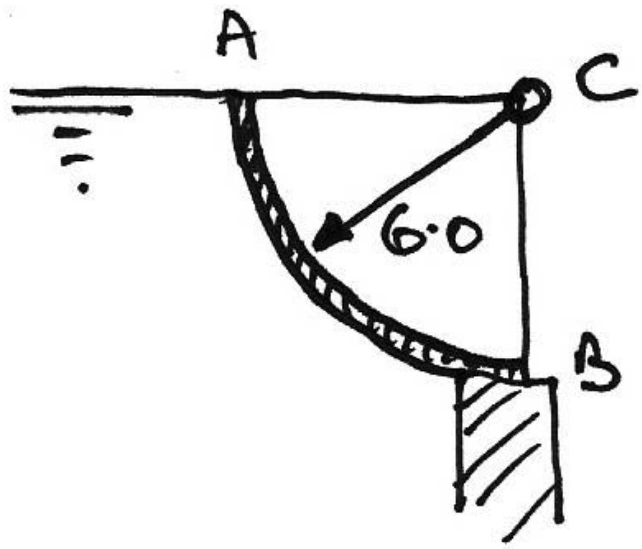
Both of these situations can be summed up with the following diagram:



Curved Surfaces – Example

Problem

Determine the resultant force and its direction on the gate shown:



Solution

The horizontal force, per metre run of the gate, is that of the surface projected onto a vertical plane of length *CB*:

$$\begin{aligned} F_x &= \rho g A_{CB} \bar{y}_{CB} \\ &= 10^3 \times 9.81 \times (6 \times 1) \times \left(\frac{6}{2}\right) \\ &= 176.6 \text{ kN} \end{aligned}$$

And this acts at a depth $h = \frac{2}{3} \cdot 6 = 4$ m from the surface. The vertical force is the weight of the imaginary water above *AB*:

$$\begin{aligned} F_y &= 10^3 \times 9.81 \left(\frac{\pi 6^2}{4} \times 1 \right) \\ &= 277.4 \text{ kN} \end{aligned}$$

In which $\pi R^2/4$ is the area of the circle quadrant. The vertical force is located at:

$$x = \frac{4R}{3\pi} = \frac{4\times 6}{3\pi} = 2.55 \text{ m}$$

to the left of line *BC*. The resultant force is thus:

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{176.6^2 + 277.4^2} \\ &= 328.8 \text{ kN} \end{aligned}$$

And acts at an angle:

$$\begin{aligned} \theta &= \tan^{-1}\frac{F_y}{F_x} \\ &= \tan^{-1}\frac{277.4}{176.6} \\ &= 57.5^\circ \end{aligned}$$

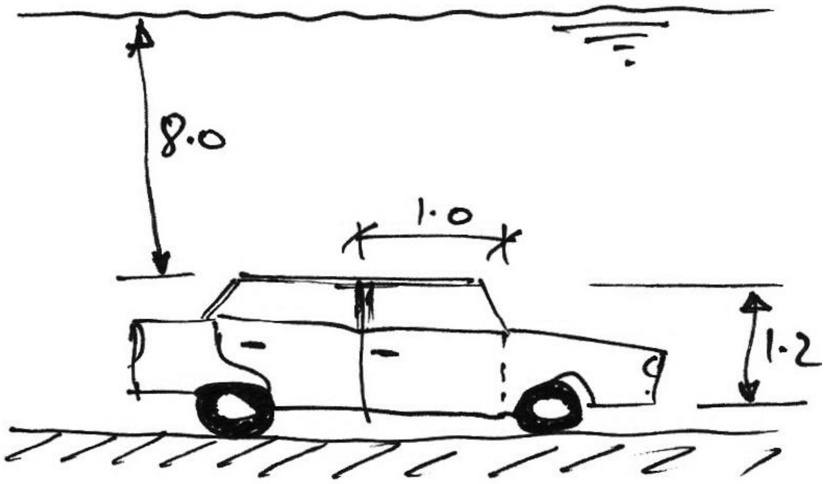
measured anticlockwise to the horizontal. The resultant passes through point *C*. Also, as the force on each infinitesimal length of the surface passes through *C*, there should be no net moment about *C*. Checking this:

$$\begin{aligned} \sum \text{Moments about } C &= 0 \\ 176.6\times 4 - 277.4\times 2.55 &= 0 \\ 706.4 - 707.4 &\approx 0 \end{aligned}$$

The error is due to rounding carried out through the calculation.

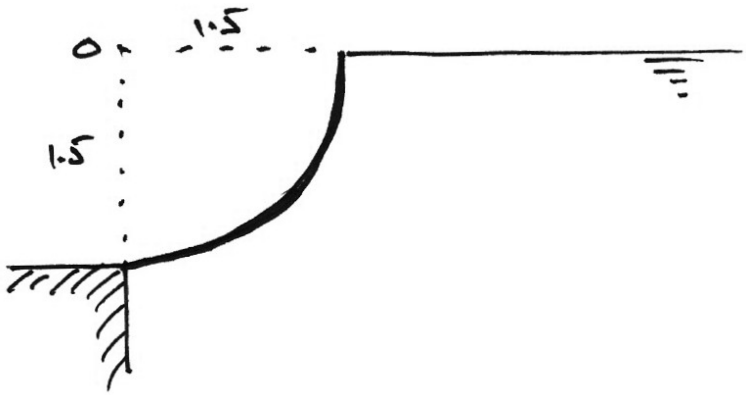
Problems – Fluid Action on Surfaces

1. You are in a car that falls into a lake to a depth as shown below. What is the moment about the hinges of the car door (1.0 × 1.2 m) due to the hydrostatic pressure? Can you open the door? What should you do?



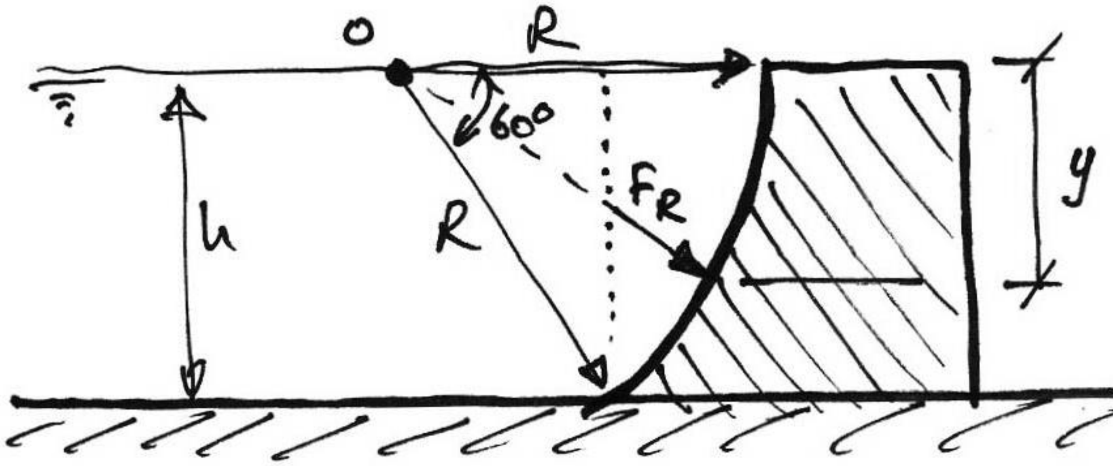
(Ans. 50.6 kNm, ?, ?)

2. A sluice gate consist of a quadrant of a circle of radius 1.5 m pivoted at its centre, O. When the water is level with the gate, calculate the magnitude and direction of the resultant hydrostatic force on the gate and the moment required to open the gate. The width of the gate is 3 m and it has a mass of 6 tonnes.



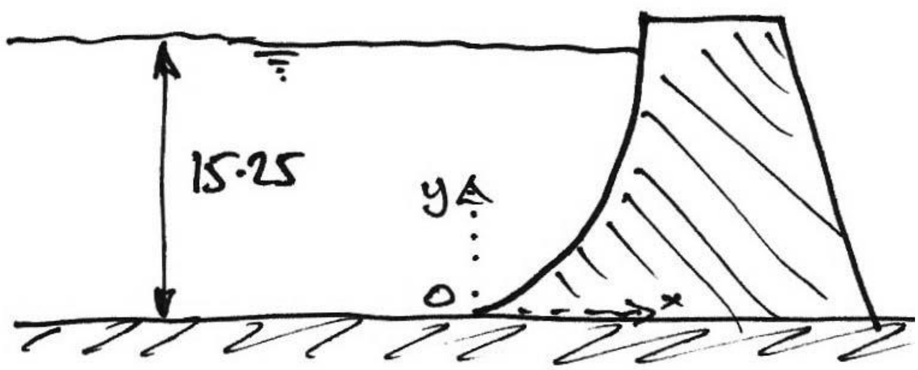
(Ans. 61.6 kN, 57°, 35.3 kNm)

3. The profile of a masonry dam is an arc of a circle, the arc having a radius of 30 m and subtending an angle of 60° at the centre of curvature which lies in the water surface. Determine: (a) the load on the dam in kN/m length; (b) the position of the line of action to this pressure.



(Ans. 4280 kN/m, 19.0 m)

4. The face of a dam is curved according to the relation $y = x^2/2.4$ where y and x are in meters, as shown in the diagram. Calculate the resultant force on each metre run of the dam. Determine the position at which the line of action of the resultant force passes through the bottom of the dam.



(Ans. 1920 kN, 14.15 m)

MODULE 2

General Concepts

Introduction

Hydrostatics involves only a few variables: ρ , *g*, and *h*, and so the equations developed are relatively simple and experiment and theory closely agree. The study of fluids in motion is not as simple and accurate. The main difficulty is viscosity.

By neglecting viscosity (an ideal fluid), we do not account for the shear forces which oppose flow. Based on this, reasonably accurate and simple theories can be derived.. Using experimental results, these theories can then be calibrated by using experimental coefficients. They then inherently allow for viscosity.

As we will be dealing with liquids, we will neglect the compressibility of the liquid. This is incompressible flow. This is not a valid assumption for gases.

Classification of Flow Pattern

There are different patterns of fluid flow, usually characterized by time and distance:

- Time: A flow is *steady* if the parameters describing it (e.g. flow rate, velocity, pressure, etc.) do not change with time. Otherwise a flow is *unsteady*.

- Distance: A flow is *uniform* if the parameters describing the flow do not change with distance. In *non-uniform* flow, the parameters change from point to point along the flow.

From these definitions almost all flows will be one of:

Steady uniform flow

Discharge (i.e. flow rate, or volume per unit time) is constant with time and the cross section of the flow is also constant. Constant flow through a long straight prismatic pipe is an example.

Steady non-uniform flow

The discharge is constant with time, but the cross-section of flow changes. An example is a river with constant discharge, as the cross section of a river changes from point to point.

Unsteady uniform flow

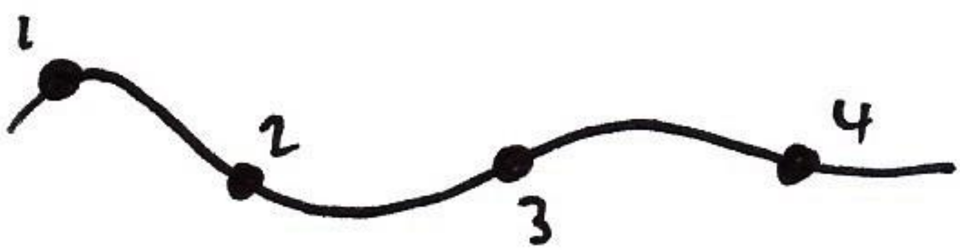
The cross-section is constant but the discharge changes with time resulting in complex flow patterns. A pressure surge in a long straight prismatic pipe is an example.

Unsteady non-uniform flow

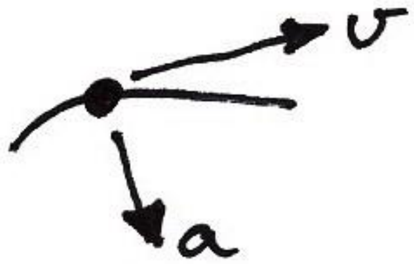
Both discharge and cross section vary. A flood wave in a river valley is an example. This is the most complex type of flow.

Visualization

To picture the motion of a fluid, we start by examining the motion of a single fluid ‘particle’ over time, or a collection of particles at one instant. This is the flow path of the particle(s), or a *streamline*:



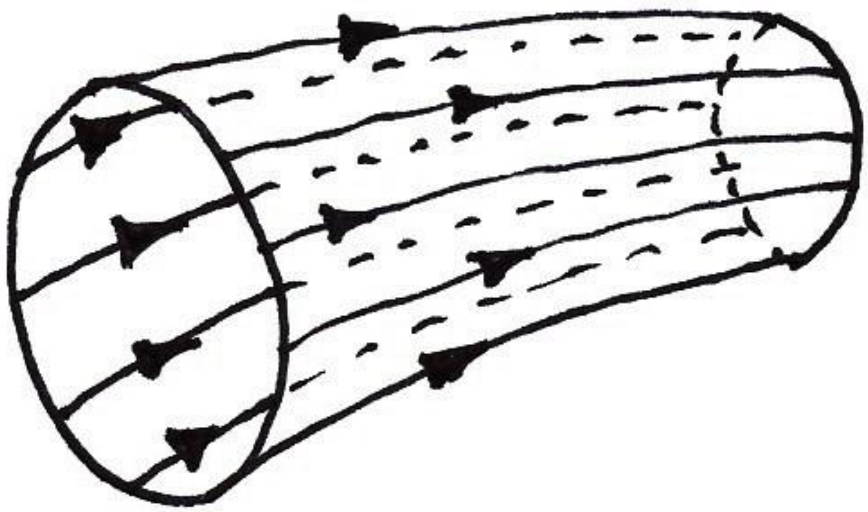
At each point, each particle has both velocity and acceleration vectors:



A streamline is thus tangential to the velocity vectors of the particles. Hence:

- there can be no flow across a streamline;
- therefore, streamlines cannot cross each other, and;
- once fluid is on a streamline it cannot leave it.

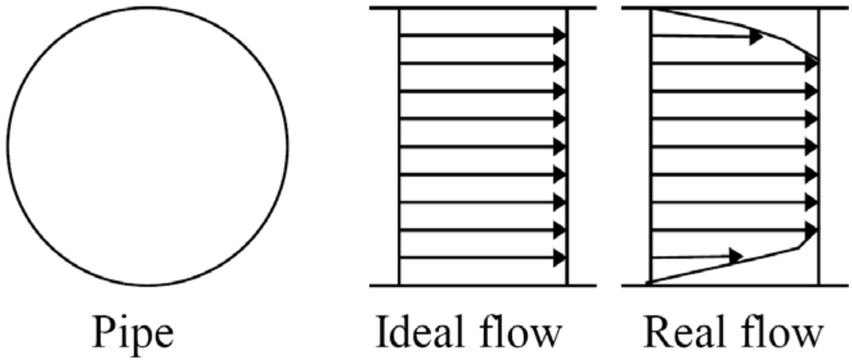
We extend this idea to a collection of paths of fluid particles to create a *streamtube*:



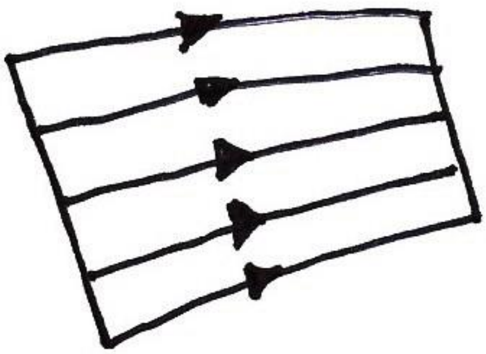
Streamlines and streamtubes are theoretical notions. In an experiment, a streakline is formed by injecting dye into a fluid in motion. A streakline therefore approximates a streamline (but is bigger because it is not an individual particle).

Dimension of Flow

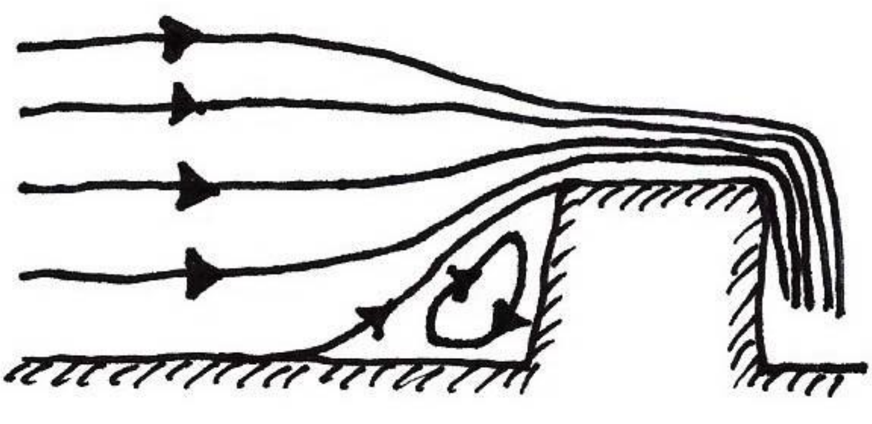
Fluid flow is in general three-dimensional in nature. Parameters of the flow can vary in the *x*, *y* and *z* directions. They can also vary with time. In practice we can reduce problems to one- or two-dimensional flow to simplify. For example:



One dimensional flow



A two-dimensional streamtube



Flow over an obstruction

Fundamental Equations

To develop equations describing fluid flow, we will work from relevant fundamental physical laws.

The Law of Conservation of Matter

Matter cannot be created nor destroyed (except in a nuclear reaction), but may be transformed by chemical reaction. In fluids we neglect chemical reactions and so we deal with the conservation of mass.

The Law of Conservation of Energy

Energy cannot be created nor destroyed, it can only be transformed from one form to another. For example, the potential energy of water in a dam is transformed to kinetic energy of water in a pipe. Though we will later talk of ‘energy losses’, this is a misnomer as none is actually lost but transformed to heat and other forms.

The Law of Conservation of Momentum

A body in motion remains in motion unless some external force acts upon it. This is Newton’s Second Law:

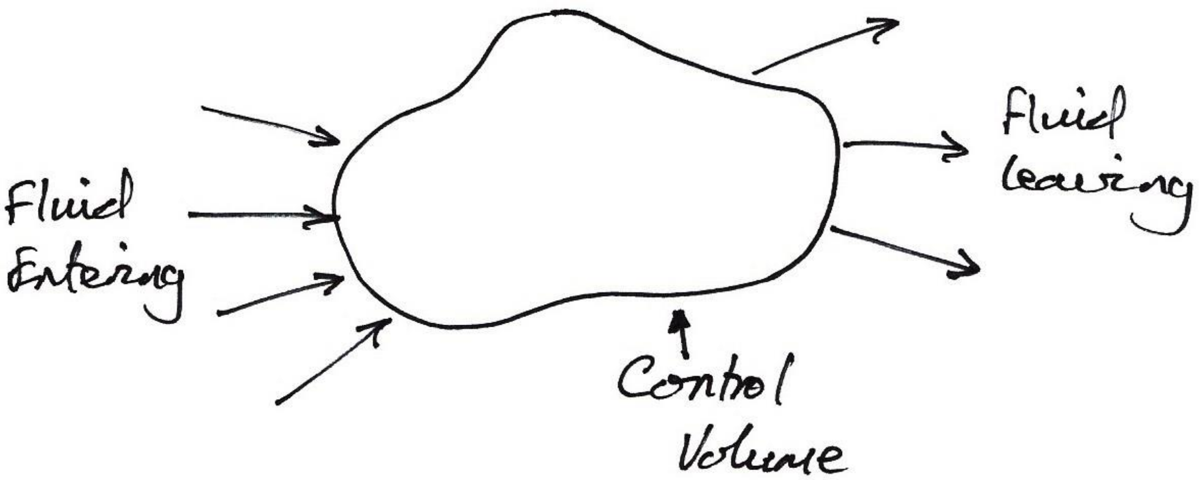
$$\text{Force} = \begin{array}{l}\text{Rate of change}\\ \text{of momentum}\end{array}$$

$$\begin{aligned} F &= \frac{d(mv)}{dt} \\ &= m\frac{dv}{dt} \\ &= ma \end{aligned}$$

To apply these laws to fluids poses a problem, since fluid is a continuum, unlike rigid bodies. Hence we use the idea of a ‘control volume’.

Control Volume

A control volume is an imaginary region within a body of flowing fluid, usually at fixed location and of a fixed size:



It can be of any size and shape so we choose shapes amenable to simple calculations. Inside the region all forces cancel out, and we can concentrate on external forces. It can be picture as a transparent pipe or tube, for example.

The Continuity Equation

Development

Applying the Law of Conservation of Mass to a control volume, we see:

$$\text{Rate of mass entering} = \text{Rate of mass leaving} + \text{Rate of mass increase}$$

For steady incompressible flow, the rate of mass increase is zero and the density of the fluid does not change. Hence:

$$\text{Rate of mass entering} = \text{Rate of mass leaving}$$

The rate of mass change can be expressed as:

$$\text{Rate of mass change} = \text{Fluid density} \times \text{Volume per second}$$

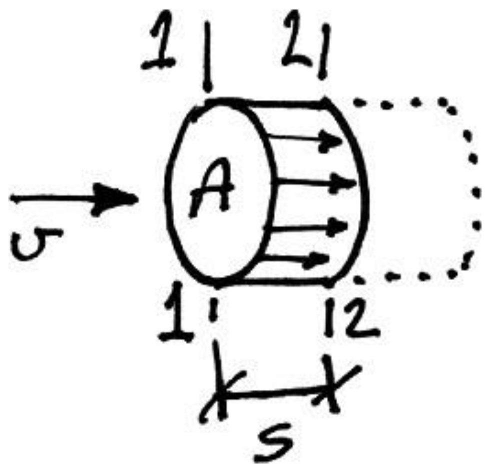
Using Q for flow rate, or volume per second (units: m^3/s, dimensions: L^3T^{-1}):

$$\rho Q_{in} = \rho Q_{out}$$

And as before, assuming that the flow is incompressible:

$$Q_{in} = Q_{out}$$

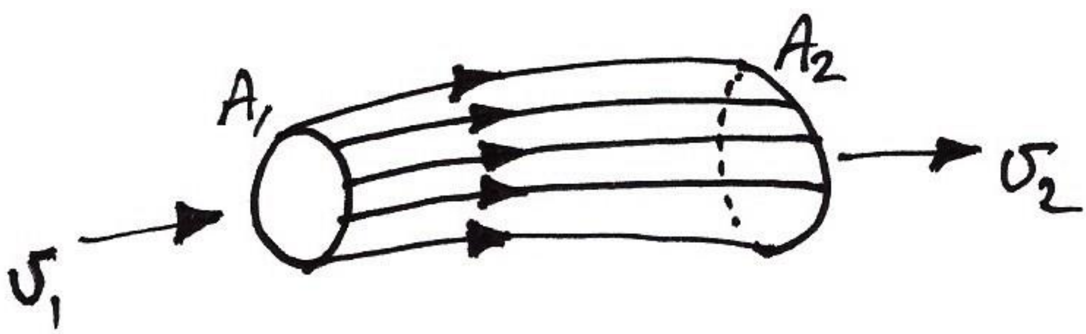
Consider a small length of streamtube:



The fluid at 1-1 moves a distance of $s = vt$ to 2-2. Therefore in 1 second it moves a distance of v. The volume moving per second is thus:

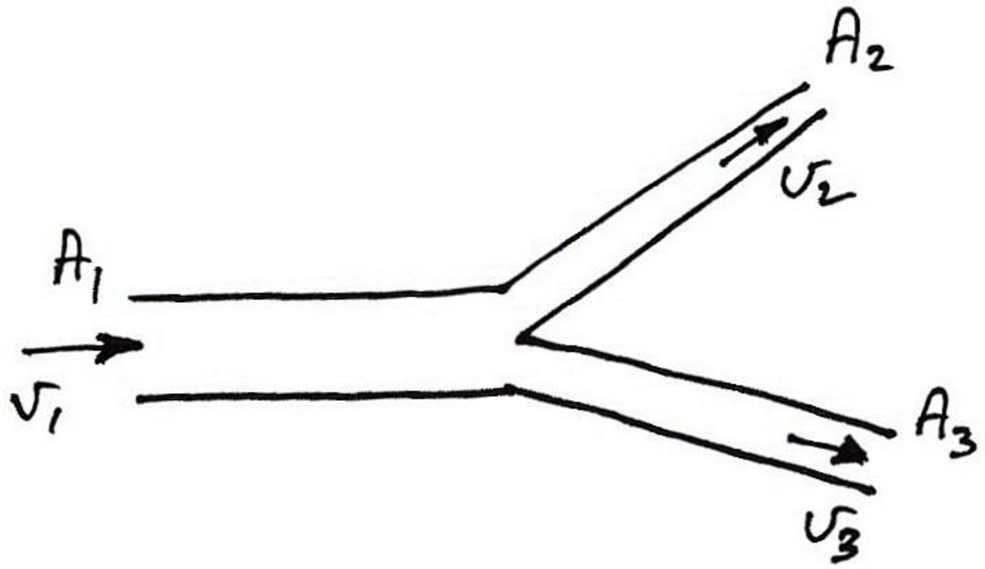
$$Q = Av$$

Thus, for an arbitrary streamtube, as shown, we have:



$$\boxed{A_1 v_1 = A_2 v_2}$$

A typical application of mass conservation is at pipe junctions:



From mass conservation we have:

$$Q_1 = Q_2 + Q_3$$
$$A_1 v_1 = A_2 v_2 + A_3 v_3$$

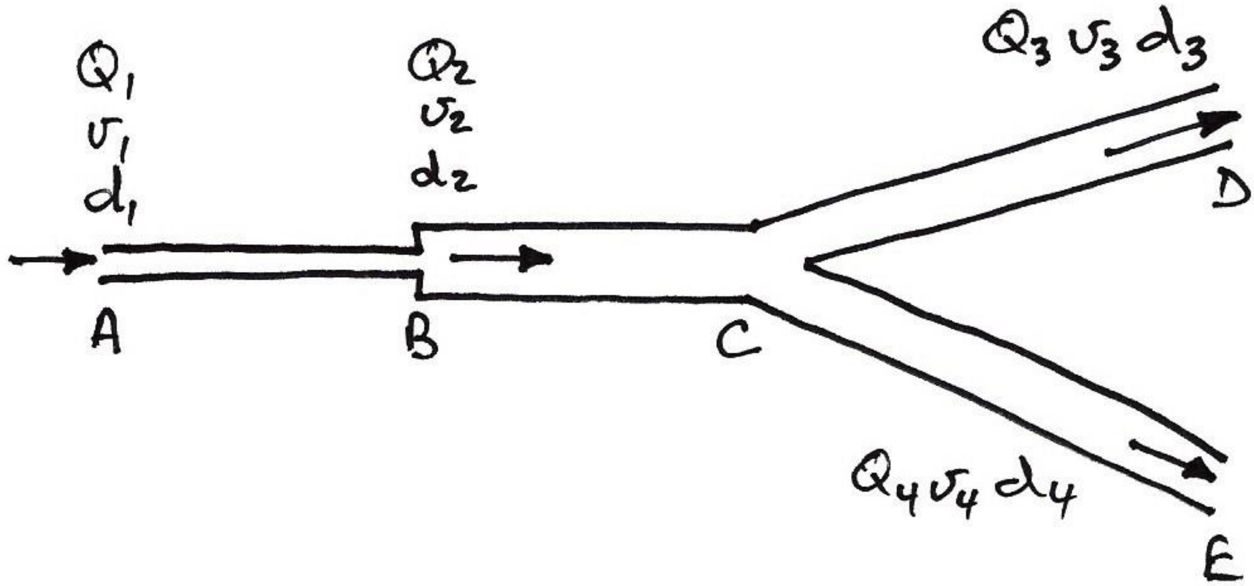
If we consider inflow to be positive and outflow negative, we have:

$$\sum_{i=1}^{\text{No. of Nodes}} A_i v_i = 0$$

Mass Conservation – Example

Problem

Water flows from point *A* to points *D* and *E* as shown. Some of the flow parameters are known, as shown in the table. Determine the unknown parameters.



Section	Diameter (mm)	Flow Rate (m^3/s)	Velocity (m/s)
AB	300	?	?
BC	600	?	1.2
CD	?	$Q_3 = 2Q_4$	1.4
CE	150	$Q_4 = 0.5Q_3$	?

Solution

From the law of mass conservation we can see:

$$Q_1 = Q_2$$

And as total inflow must equal total outflow:

$$\begin{aligned} Q_1 &= Q_{out} \\ &= Q_3 + Q_4 \\ &= Q_3 + 0.5Q_3 \\ &= 1.5Q_3 \end{aligned}$$

We must also work out the areas of the pipes, $A_i = \frac{\pi d_i^2}{4}$. Hence:

$A_1 = 0.0707\ \text{m}^3$ $A_2 = 0.2827\ \text{m}^3$ $A_4 = 0.0177\ \text{m}^3$

Starting with our basic equation, $Q = Av$, we can only solve for Q_2 from the table:

$$\begin{aligned} Q_2 &= (0.2827)(1.2) \\ &= 0.3393\ \text{m}^3/s \end{aligned}$$

We know that $Q_1 = Q_2$ and so we can now calculate Q_3 from previous:

$$Q_1 = 1.5Q_3$$
$$Q_3 = \frac{Q_1}{1.5} = \frac{0.3393}{1.5} = 0.2262\ \text{m}^3/\text{s}$$

And so,

$$Q_4 = \frac{Q_3}{2} = \frac{0.2262}{2} = 0.1131 \text{ m}^3\text{/s}$$

Thus we have all the flows. The unknown velocities are:

$$v_1 = \frac{Q_1}{A_1} = \frac{0.3393}{0.0707} = 4.8 \text{ m/s}$$

$$v_4 = \frac{Q_4}{A_4} = \frac{0.1131}{0.0177} = 6.4 \text{ m/s}$$

And lastly, the diameter of pipe *CD* is:

$$A_3 = \frac{Q_3}{v_3} = \frac{0.2262}{1.4} = 0.1616 \text{ m}^2$$

$$d_3 = \sqrt{\frac{4A_3}{\pi}} = 0.454 \text{ m}$$

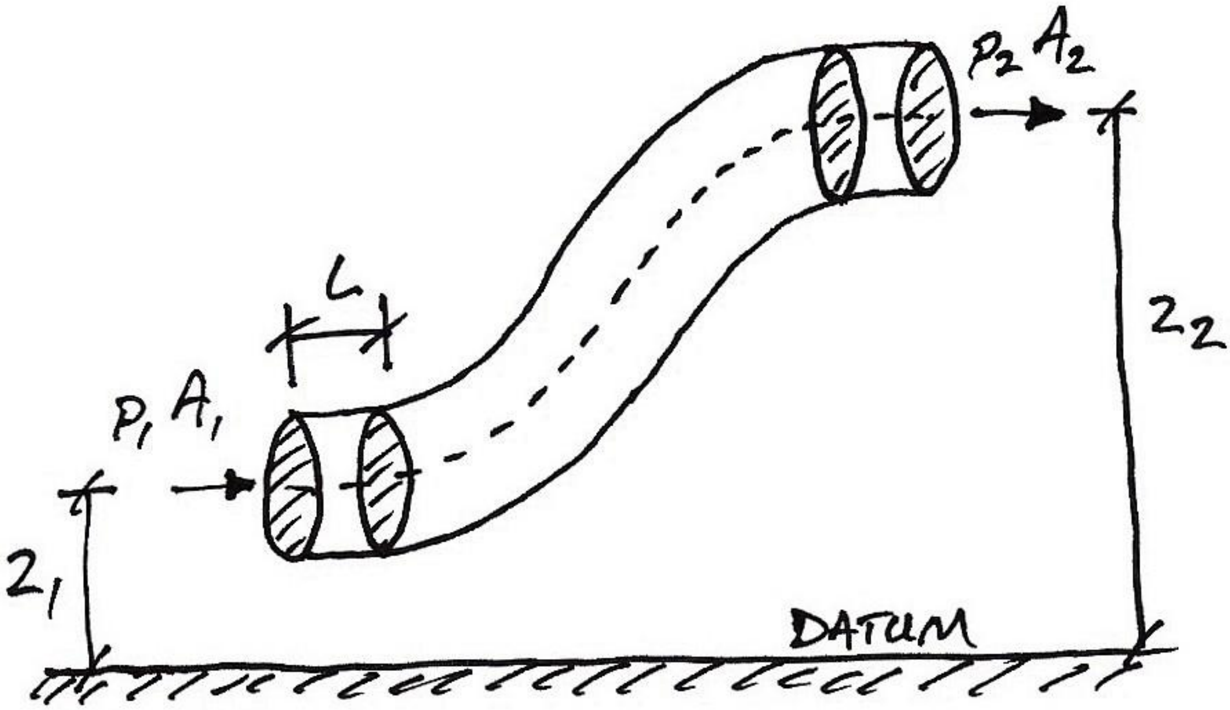
And so it is likely to be a 450 mm $\varnothing$ pipe.

Note that in a problem such as this the individual calculations do not pose a problem. It is the strategy used to solve it that is key. In this problem, we started from some knowns and one equation. Even though we couldn't see all the way to the end from Step 1, with each new calculation another possibility opened up. This is the 'art of problem solving' and it can only be learned by practice!

The Energy Equation

Development

We apply the Law of Conservation of Energy to a control volume. To do so, we must identify the forms of energy in the control volume. Consider the following system:



The forms of energy in this system are:

- Pressure energy:

  The pressure in a fluid also does work by generating force on a cross section which then moves through a distance. This is energy since work is energy.
- Kinetic energy:

  This is due to the motion of the mass of fluid.
- Potential energy:

  This is due to the height above an arbitrary datum.

Pressure Energy

The combination of flow and pressure gives us work. The pressure results in a force on the cross section which moves through a distance L in time δt . Hence the pressure energy is the work done on a mass of fluid entering the system, which is:

$$m = \rho_1 A_1 L$$

Be careful to distinguish the density ρ and the pressure p.

And so the pressure energy at the entry is:

$$\text{PrE} = pAL = p_1 A_1 L$$

Kinetic Energy

From classical physics, the kinetic energy of the mass entering is:

$$\text{KE} = \frac{1}{2} m v^2 = \frac{1}{2} \rho_1 A_1 L v_1^2$$

Potential Energy

The potential energy of the mass entering, due to the height above the datum is:

$$\text{PE} = mgz = \rho_1 A_1 L g z_1$$

Total Energy

The total energy at the entry to the system is just the sum:

$$H_1^* = p_1 A_1 L + \frac{1}{2} \rho_1 A_1 L v_1^2 + \rho_1 A_1 L g z_1$$

Final Form

It is more usual to consider the energy per unit weight, and so we divide through by $mg = \rho_1 g A_1 L$:

$$
\begin{aligned}
H_1 &= \frac{H_1^*}{\rho_1 g A_1 L} \\
&= \frac{p_1 A_1 L}{\rho_1 g A_1 L} + \frac{1}{2}\frac{\rho_1 A_1 L v_1^2}{\rho_1 g A_1 L} + \frac{\rho_1 A_1 L g z_1}{\rho_1 g A_1 L} \\
&= \frac{p_1}{\rho_1 g} + \frac{v_1^2}{2g} + z_1
\end{aligned}
$$

Similarly, the energy per unit weight leaving the system is:

$$H_2 = \frac{p_2}{\rho_2 g} + \frac{v_2^2}{2g} + z_2$$

Also, the energy entering must equal the energy leaving as we assume the energy cannot change. Also, assuming incompressibility, the density does not change:

$$
\begin{gathered}
H_1 = H_2 \\
\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2
\end{gathered}
$$

And so we have Bernoulli's Equation:

$$\boxed{\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 = H = \text{constant}}$$

Comments

From Bernoulli's Equation we note several important aspects:

1. It is assumed that there is no energy taken from or given to the fluid between the entry and exit. This implies the fluid is frictionless as friction generates heat energy which would be an energy loss. It also implies that there is no energy added, say by a pump for example.

2. Each term of the equation has dimensions of length, L, and units of metres. Therefore each term is known as a head:
   - Pressure head: $\frac{p}{\rho g}$;
   - Kinetic or velocity head: $\frac{v^2}{2g}$;
   - Potential or elevation head: z.

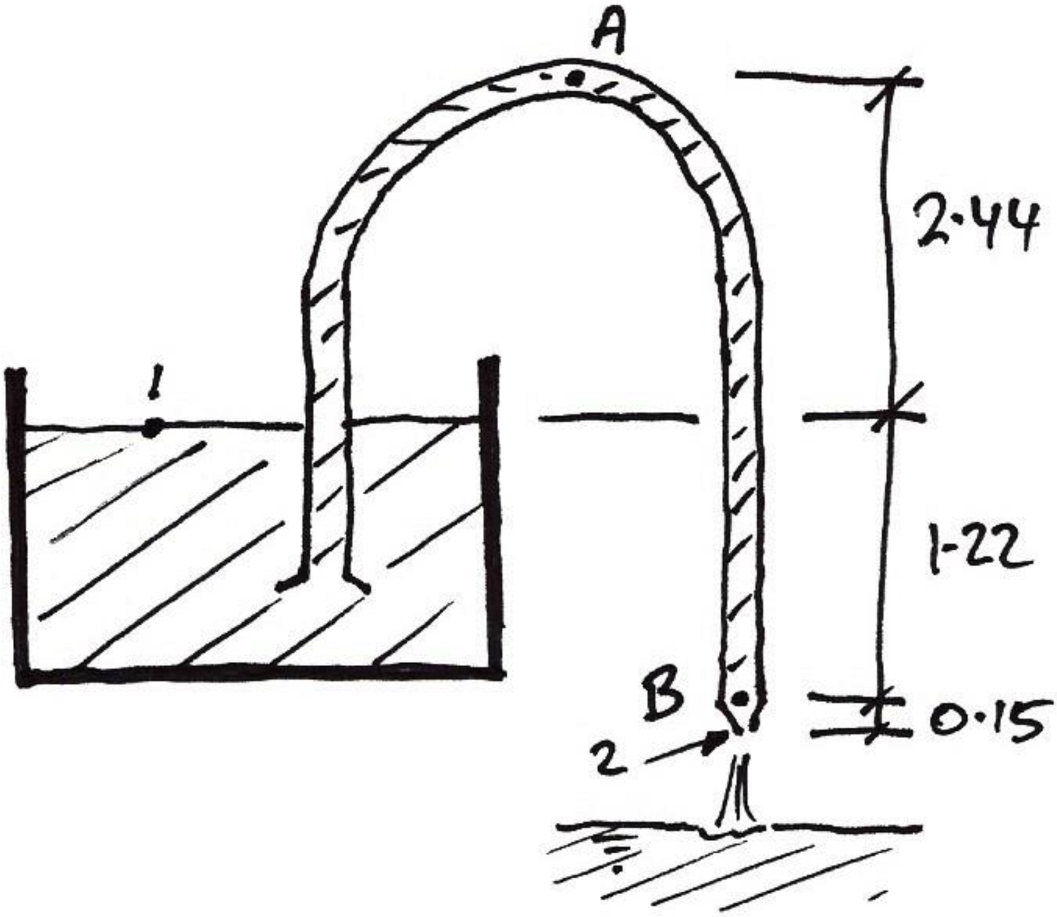
3. The streamtube must have very small dimensions compared to the heights above the datum. Otherwise the height to the top of a cross-section would be different to the height to the bottom of a cross-section. Therefore, Bernoulli's Equation strictly only applies to streamlines.

We have derived the equation from energy considerations. It can also be derived by force considerations upon an elemental piece of fluid.

Energy Equation – Example

Problem

For the siphon shown, determine the discharge and pressure heads at *A* and *B* given that the pipe diameter is 200 mm and the nozzle diameter is 150 mm. You may neglect friction in the pipe.



Solution

To find the discharge (or flow) apply Bernoulli's Equation along the streamline connecting points 1 and 2. To do this note:

- Both p_1 and p_2 are at atmospheric pressure and are taken to be zero;
- v_1 is essentially zero.

$$\underbrace{\left(\frac{p_1}{\rho g}\right)}_{=0} + \underbrace{\left(\frac{v_1^2}{2g}\right)}_{=0} + z_1 = \underbrace{\left(\frac{p_2}{\rho g}\right)}_{=0} + \frac{v_2^2}{2g} + z_2$$

$$z_1 - z_2 = \frac{v_2^2}{2g}$$

Hence, from the figure:

$$1.22 + 0.15 = \frac{v_2^2}{2 \times 9.81}$$

$$v_2 = 5.18 \text{ m/s}$$

And using continuity:

$$\begin{aligned} Q_2 &= A_2 v_2 \\ &= \frac{\pi (0.15)^2}{4} \cdot 5.18 \\ &= 0.092 \text{ m}^3 / s \end{aligned}$$

For the pressure head at A, apply Bernoulli's equation from point 1 to A:

$$\underbrace{\left(\frac{p_1}{\rho g}\right)}_{=0} + \underbrace{\left(\frac{v_1^2}{2g}\right)}_{=0} + z_1 = \frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A$$

Hence:

$$\frac{p_A}{\rho g} = \left(z_1 - z_A\right) - \frac{v_A^2}{2g}$$

Again using continuity between point 2 and *A* and the diameter of the pipe at *A*:

$$\begin{aligned} Q_A &= Q_2 \\ A_A v_A &= 0.092 \\ \frac{\pi\left(0.2\right)^2}{4} \cdot v_A &= 0.092 \\ v_A &= 2.93 \text{ m/s} \end{aligned}$$

Hence the kinetic head at *A* is just $\frac{v_A^2}{2g} = 0.44 \text{ m}$, and so:

$$\begin{aligned} \frac{p_A}{\rho g} &= -2.44 - 0.44 \\ &= -2.88 \text{ m} \end{aligned}$$

This is negative gauge pressure indicating suction. However, it is still a positive absolute pressure.

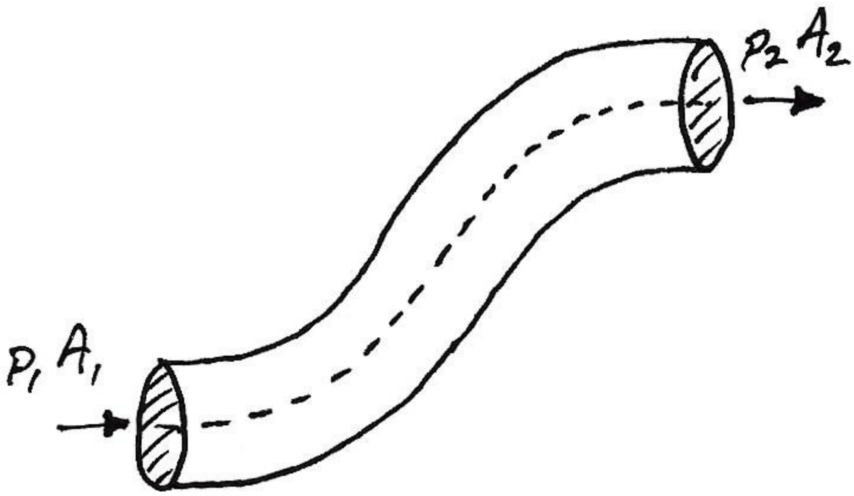
Similarly to *A*, at *B* we have, $v_B = v_A$ and $z_1 - z_B = 1.22 \text{ m}$ and so:

$$\begin{aligned} \frac{p_B}{\rho g} &= \left(z_1 - z_B\right) - \frac{v_B^2}{2g} \\ &= 1.22 - 0.44 \\ &= 0.78 \text{ m} \end{aligned}$$

The Momentum Equation

Development

We consider again a general streamtube:



In a given time interval, δt , we have:

$$\text{momentum entering} = \rho\, Q_1\, \delta t\, v_1$$
$$\text{momentum leaving} = \rho\, Q_2\, \delta t\, v_2$$

From continuity we know $Q = Q_1 = Q_2$. Thus the force required giving the change in momentum between the entry and exit is, from Newton's Second Law:

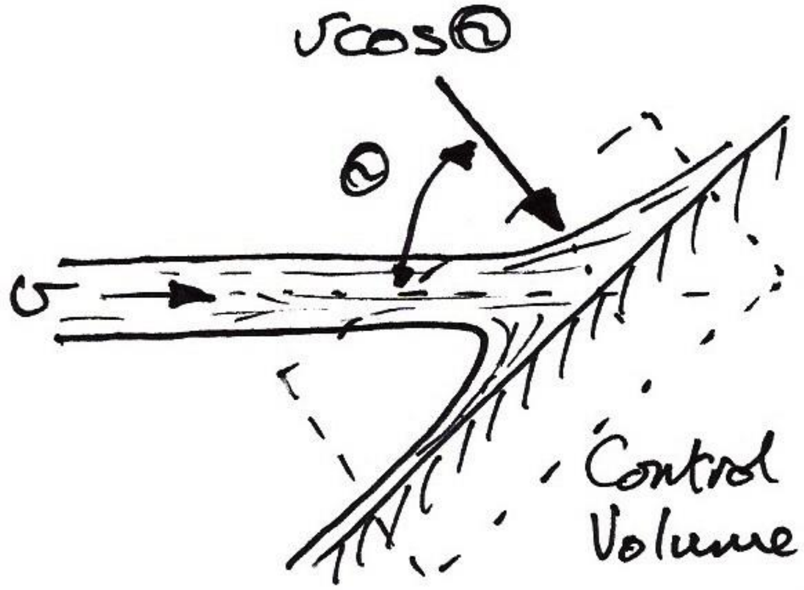
$$F = \frac{d(mv)}{dt}$$

$$F = \frac{\rho\, Q \delta t (v_2 - v_1)}{\delta t}$$
$$= \rho\, Q (v_2 - v_1)$$

This is the force acting on a fluid element in the direction of motion. The fluid exerts an equal but opposite reaction to its surroundings.

Application – Fluid Striking a Flat Surface

Consider the jet of fluid striking the surface as shown:



The velocity of the fluid normal to the surface is:

$$v_{normal} = v\cos\theta$$

This must be zero since there is no relative motion at the surface. This then is also the change in velocity that occurs normal to the surface. Also, the mass flow entering the control volume is:

$$\rho Q = \rho Av$$

Hence:

$$
\begin{aligned}
F &= \frac{d(mv)}{dt} \\
&= (\rho A v)(v\cos\theta) \\
&= \rho A v^2 \cos\theta
\end{aligned}
$$

And if the plate is perpendicular to the flow then:

$$F = \rho A v^2$$

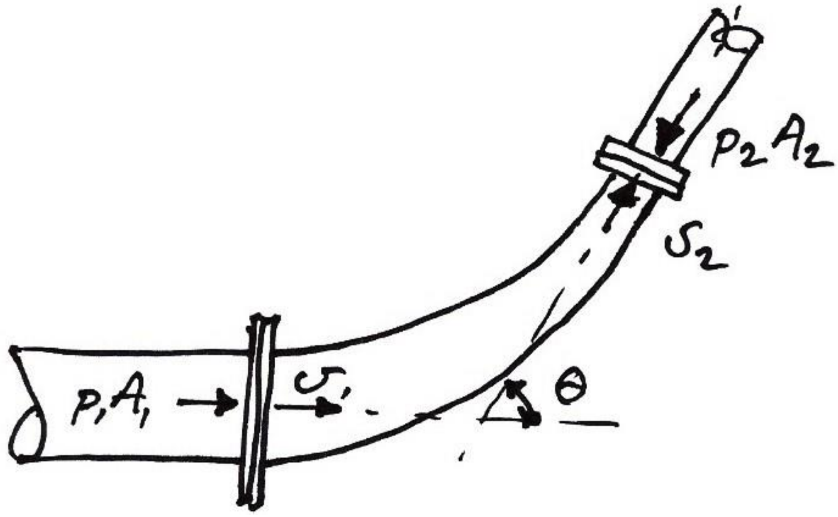
Notice that the force exerted by the fluid on the surface is proportional to the velocity squared. This is important for wind loading on buildings. For example, the old wind loading code CP3: Chapter V gives as the pressure exerted by wind as:

$$q = 0.613 v_s^2 \qquad (\mathrm{N/m^2})$$

In which v_s is the design wind speed read from maps and modified to take account of relevant factors such as location and surroundings.

Application – Flow around a bend in a pipe

Consider the flow around the bend shown below. We neglect changes in elevation and consider the control volume as the fluid between the two pipe joins.



The net external force on the control volume fluid in the *x*-direction is:

$$p_1A_1 - p_2A_2\cos\theta + F_x$$

In which F_x is the force on the fluid by the pipe bend (making it 'go around the corner'). The above net force must be equal to the change in momentum, which is:

$$\rho Q\left(v_2\cos\theta - v_1\right)$$

Hence:

$$p_1A_1 - p_2A_2\cos\theta + F_x = \rho Q\left(v_2\cos\theta - v_1\right)$$

$$F_x = \rho Q\left(v_2\cos\theta - v_1\right) - p_1A_1 + p_2A_2\cos\theta$$

$$= \left(\rho Q v_2 + p_2A_2\right)\cos\theta - \left(\rho Q v_1 + p_1A_1\right)$$

Similarly, for the *y*-direction we have:

–

$$
\begin{aligned}
p_2 A_2 \sin\theta + F_y &= \rho Q\left(v_2 \sin\theta - 0\right) \\
F_y &= \rho Q\left(v_2 \sin\theta - 0\right) + p_2 A_2 \sin\theta \\
&= \left(\rho Q v_2 + p_2 A_2\right)\sin\theta
\end{aligned}
$$

The resultant is:

$$F = \sqrt{F_x^2 + F_y^2}$$

And which acts at an angle of:

$$\theta = \tan^{-1}\frac{F_y}{F_x}$$

This is the force and direction of the bend on the fluid. The bend itself must then be supported for this force. In practice a manhole is built at a bend, or else a thrust block is used to support the pipe bend.

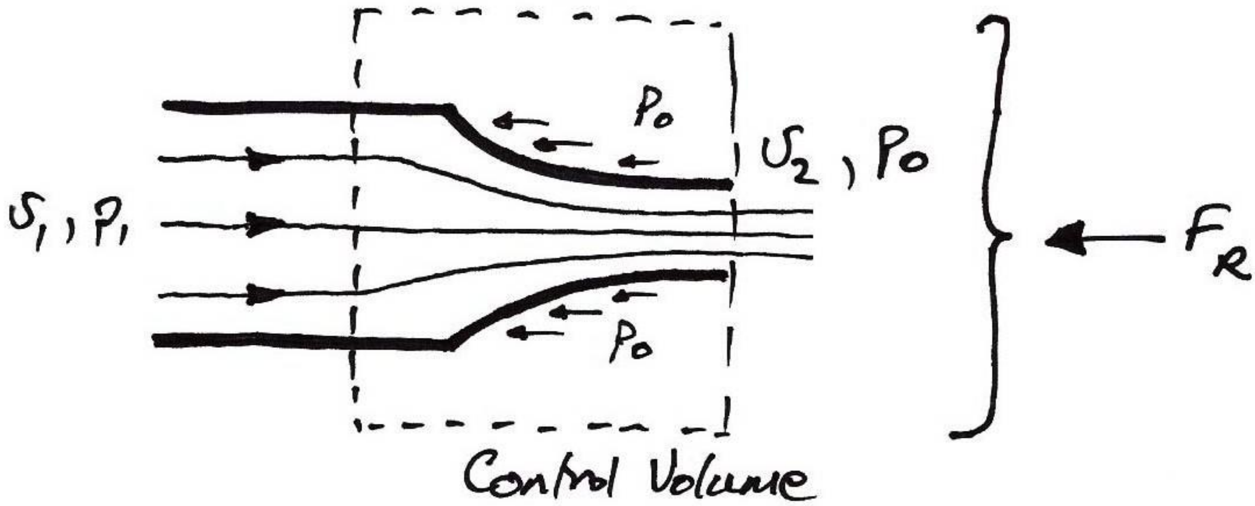
Application – Force exerted by a firehose

Problem

A firehose discharges 5 l/s. The nozzle inlet and outlet diameters are 75 and 25 mm respectively. Calculate the force required to hold the hose in place.

Solution

The control volume is taken as shown:



There are three forces in the *x*-direction:

- The reaction force F_R provided by the fireman;
- Pressure forces F_P: p_1A_1 at the left side and p_0A_0 at the right hand side;
- The momentum force F_M caused by the change in velocity.

So we have:

$$F_M = F_P + F_R$$

The momentum force is:

$$F_M = \rho Q\left(v_2 - v_1\right)$$

Therefore, we need to establish the velocities from continuity:

$$v_1 = \frac{Q}{A_1} = \frac{5\times 10^{-3}}{\pi\left(0.075\right)^2/4}$$
$$= 1.13 \text{ m/s}$$

And

$$v_2 = \frac{5\times 10^{-3}}{\pi\left(0.025\right)^2/4}$$
$$= 10.19 \text{ m/s}$$

Hence:

$$\begin{aligned} F_M &= \rho Q\left(v_2 - v_1\right) \\ &= 10^3\left(5\times 10^{-3}\right)\left(10.19 - 1.13\right) \\ &= 45 \text{ N} \end{aligned}$$

The pressure force is:

$$F_P = p_1 A_1 - p_0 A_0$$

If we consider gauge pressure only, the $p_0 = 0$ and we must only find p_1. Using Bernoulli's Equation between the left and right side of the control volume:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \underbrace{\left(\frac{p_0}{\rho g} \right)}_{=0} + \frac{v_0^2}{2g}$$

Thus:

$$\begin{aligned} p_1 &= \left(\frac{\rho}{2} \right) \left(v_1^2 - v_0^2 \right) \\ &= \left(\frac{10^3}{2} \right) \left(10.19^2 - 1.13^2 \right) \\ &= 51.28 \text{ kN/m}^2 \end{aligned}$$

Hence

$$\begin{aligned} F_P &= p_1 A_1 - p_0 A_0 \\ &= \left(51.28 \times 10^3 \right) \left(\frac{\pi \left(0.075 \right)^2}{4} \right) - 0 \\ &= 226 \text{ N} \end{aligned}$$

Hence the reaction force is:

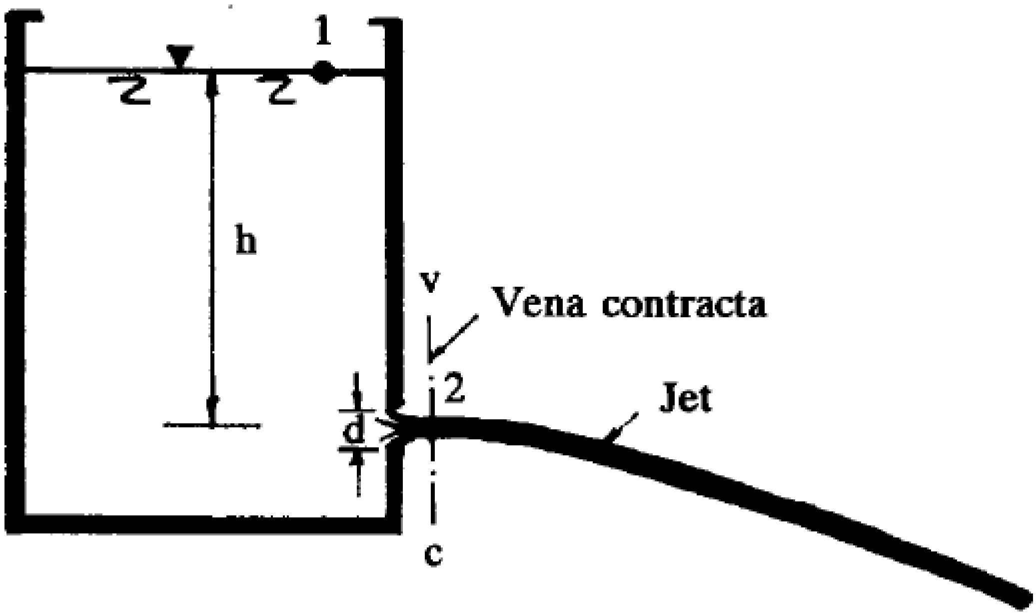
$$\begin{aligned} F_R &= F_M - F_P \\ &= 45 - 226 \\ &= -181 \text{ N} \end{aligned}$$

This is about a fifth of an average body weight – not inconsequential.

MODULE 3

Flow Measurement – Small Orifices

Consider the following tank discharge through a small opening below its surface:



If the head is practically constant across the diameter of the orifice ($h > d$) then, using the energy equation:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + h = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + 0$$

With both pressures atmospheric and taking $v_1 = 0$ we have:

$$h = \frac{v_2^2}{2g}$$

And so the velocity through the orifice is:

$$v_2 = \sqrt{2gh}$$

This is Torricelli's Theorem and represents the theoretical velocity through the orifice. Measured velocities never quite match this theoretical velocity and so we introduce a *coefficient of velocity*, C_v, to get:

$$v_{actual} = C_v\sqrt{2gh}$$

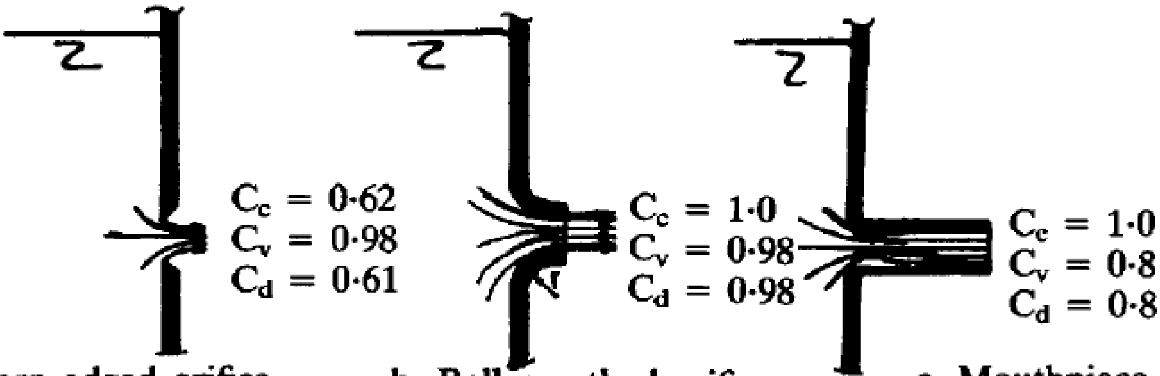
Also, due to viscosity the area of the jet may not be the same as that of the orifice and so we introduce a *coefficient of contraction*, C_c:

$$C_c = \frac{\text{Area of jet}}{\text{Area of orifice}}$$

Lastly, the discharge through the orifice is then:

$$\begin{aligned} Q &= Av \\ &= (C_c a)\left(C_v\sqrt{2gh}\right) \\ &= C_d a\sqrt{2gh} \end{aligned}$$

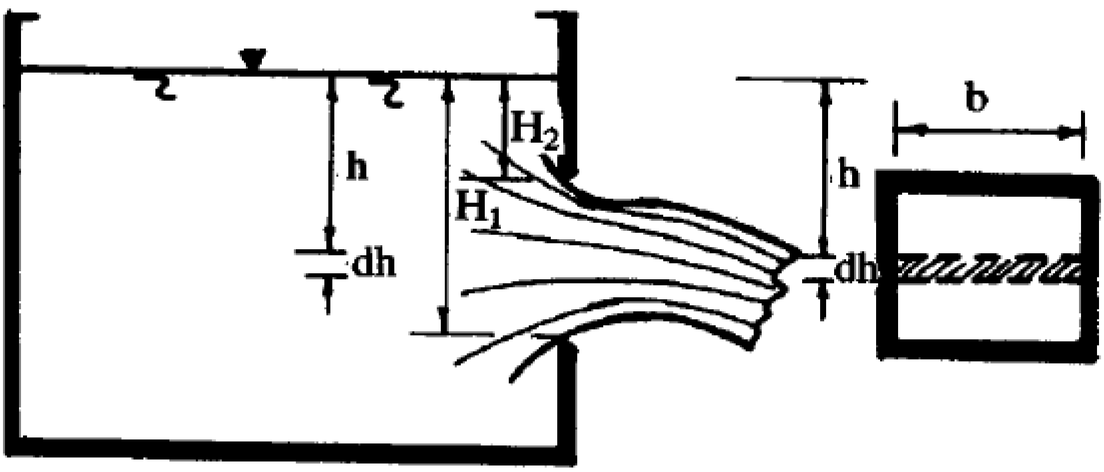
In which C_d is the coefficient of discharge and is equal to C_cC_v. For some typical orifices and mouthpieces values of the coefficient are:



a. Sharp edged orifice b. Bell mouthed orifice c. Mouthpiece

Flow Measurement – Large Orifices

When studying small orifices we assumed that the head was effectively constant across the orifice. With large openings this assumption is not valid. Consider the following opening:



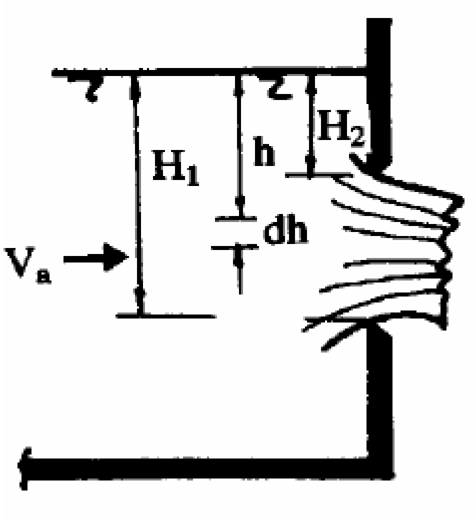
To proceed, we consider the infinitesimal rectangular strip of area $b \cdot dh$ at depth *h*. The velocity through this area is $\sqrt{2gh}$ and the infinitesimal discharge through it is:

$$dq = C_d\, b\, dh \sqrt{2gh}$$

Thus the total discharge through the opening is the sum of all such infinitesimal discharges:

$$\begin{aligned} Q &= \int dq \\ &= C_d b \sqrt{2g} \int_{H_2}^{H_1} \sqrt{h}\, dh \\ &= \frac{2}{3} C_d\, b \sqrt{2g} \left(H_1^{3/2} - H_2^{3/2} \right) \end{aligned}$$

Large openings are common in civil engineering hydraulics, for example in weirs. But in such cases the fluid has a velocity (V_a) approaching the large orifice:



Using the energy equation:

$$\frac{V_a^2}{2g} + h = \frac{v_{jet}^2}{2g}$$

Hence:

$$v_{jet} = \sqrt{2gh + V_a^2}$$
$$= \sqrt{2g\left(h + V_a^2/2g\right)}$$

In which each term in the brackets is a head. Given the velocity we can find the discharge through the strip to be:

$$dq = C_d\, b\, dh \sqrt{2g\left(h + V_a^2/2g\right)}$$

And so the total discharge is:

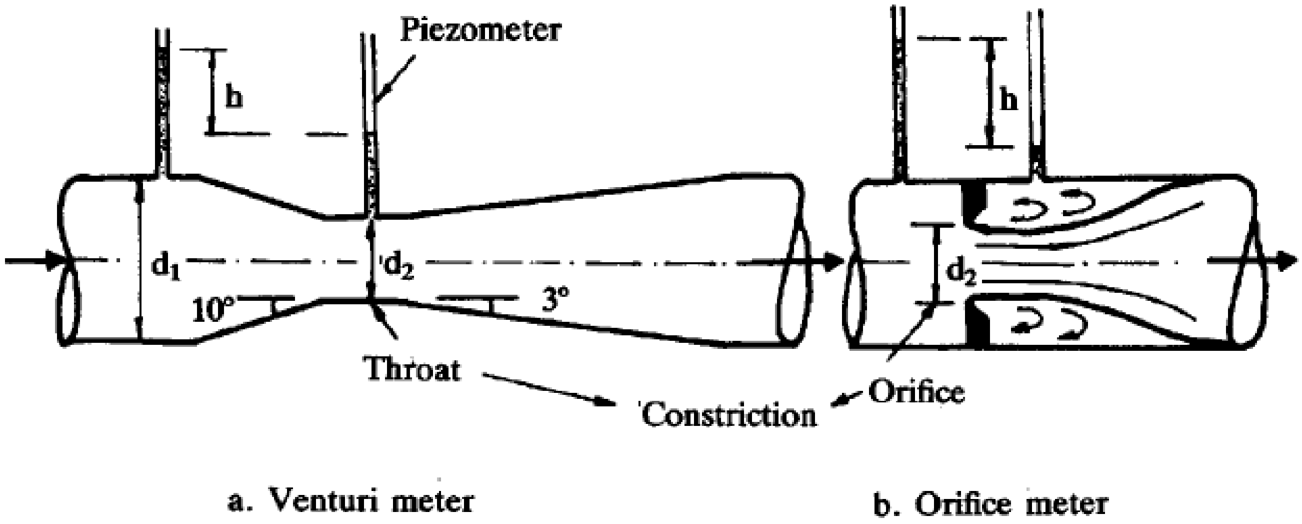
$$Q = \int dq$$

$$= C_d b\sqrt{2g}\int_{H_2}^{H_1} \sqrt{h + V_a^2/2g}\; dh$$

$$= \frac{2}{3} C_d b\sqrt{2g}\left[\left(H_1 + V_a^2/2g\right)^{3/2} - \left(H_2 + V_a^2/2g\right)^{3/2}\right]$$

Discharge Measurement in Pipelines

We consider two kinds of meters based on constricting the flow: the Venturi meter and the Orifice meter, as shown.



The constriction in these meters causes a difference in pressure between points 1 and 2, and it is this pressure difference that enables the discharge to be measured. Applying the energy equation between the inlet and the constriction:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

Thus the difference in height in the piezometer is:

$$h = \frac{p_1 - p_2}{\rho g} = \frac{v_2^2 - v_1^2}{2g}$$

And from continuity $Q = A_1 v_1 = A_2 v_2$, and using $k = A_1 / A_2$ we get:

$$v_2^2 = \left(A_1^2 / A_2^2 \right) v_1^2 = k^2 v_1^2$$

And also:

$$v_1^2 = 2gh + v_2^2$$

Which after substituting for v_2^2 and rearranging gives:

$$v_1 = \sqrt{\frac{2gh}{k^2 - 1}}$$

Hence the discharge is:

$$\begin{aligned} Q &= A_1 v_1 \\ &= A_1 \sqrt{\frac{2gh}{k^2 - 1}} \end{aligned}$$

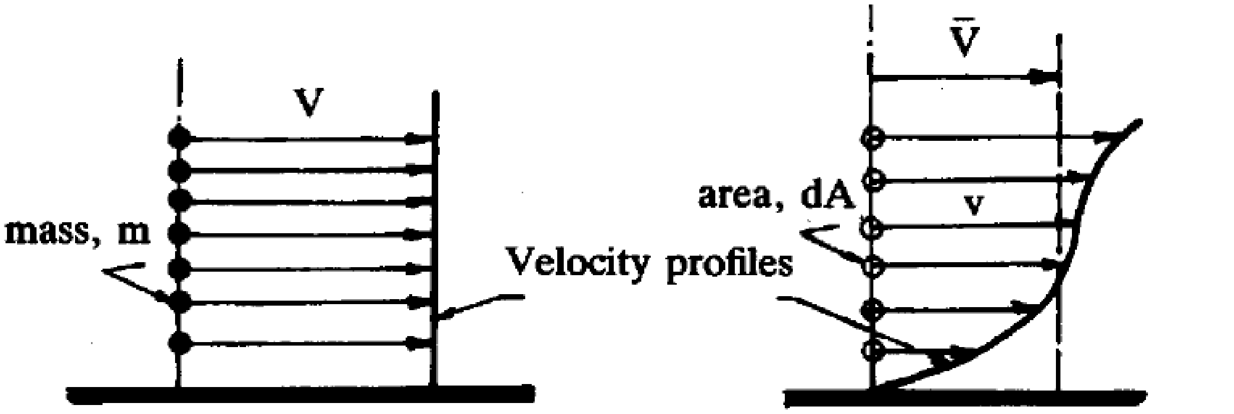
This equation neglects all losses. The actual discharge requires the introduction of the coefficient of discharge, C_d :

$$Q_{actual} = C_d A_1 \sqrt{\frac{2gh}{k^2 - 1}}$$

For properly designed Venturi meters, C_d is about 0.97 to 0.99 but for the Orifice meter it is much lower at about 0.65.

Velocity and Momentum Factors

The velocity and momentum terms in the energy and momentum equations (and in any resultant developments) have assumed uniform flow and thus require modification due to velocity variations:



a. Uniform distribution   b. Nonuniform distribution

We require a factor that accounts for the real velocity profile and so we equate kinetic energies. For the true profile the mass passing through a small area is $\rho v\, dA$. Hence the kinetic energy passing through this small area is $\frac{1}{2}(\rho v\, dA)v^2$ and so the total energy is:

$$\frac{1}{2}\rho\int v^3 dA$$

With an imaginary uniform flow of the average velocity $\bar{V}$, the total energy is $\alpha\frac{1}{2}\rho\bar{V}^3 A$ in which α is the velocity correction factor. Hence:

$$\alpha\frac{1}{2}\rho\bar{V}^3 A = \frac{1}{2}\rho\int v^3 dA$$

And so

$$\alpha = \frac{1}{A}\int\left(\frac{v}{\overline{V}}\right)^3 dA$$

Usual values for α are 1.03 to 1.3 for turbulent flows and 2 for laminar flows.

The momentum correction factor follows a similar idea and is:

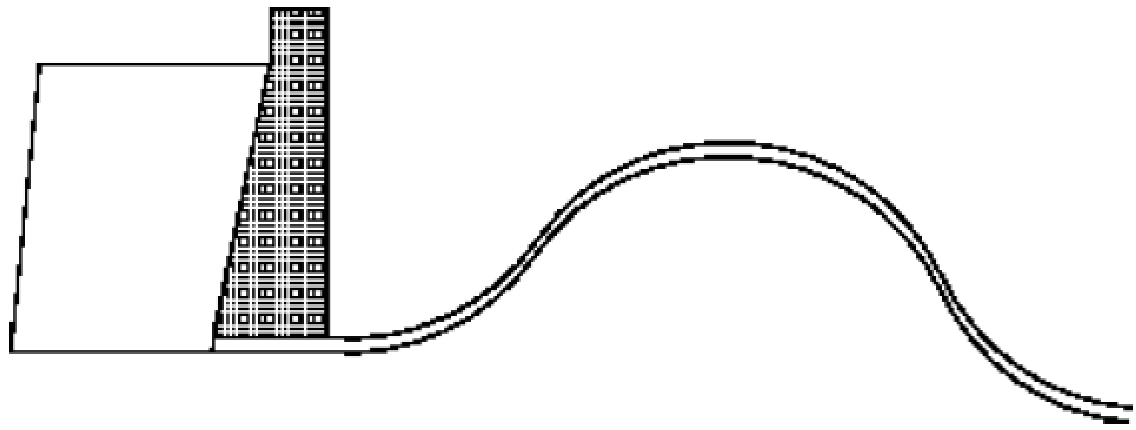
$$\beta = \frac{1}{A}\int\left(\frac{v}{\overline{V}}\right)^2 dA$$

Its values are lower than α .

Both correction factors are usually close to unity and are usually ignored, but this is not always the case.

Accounting for Energy Losses

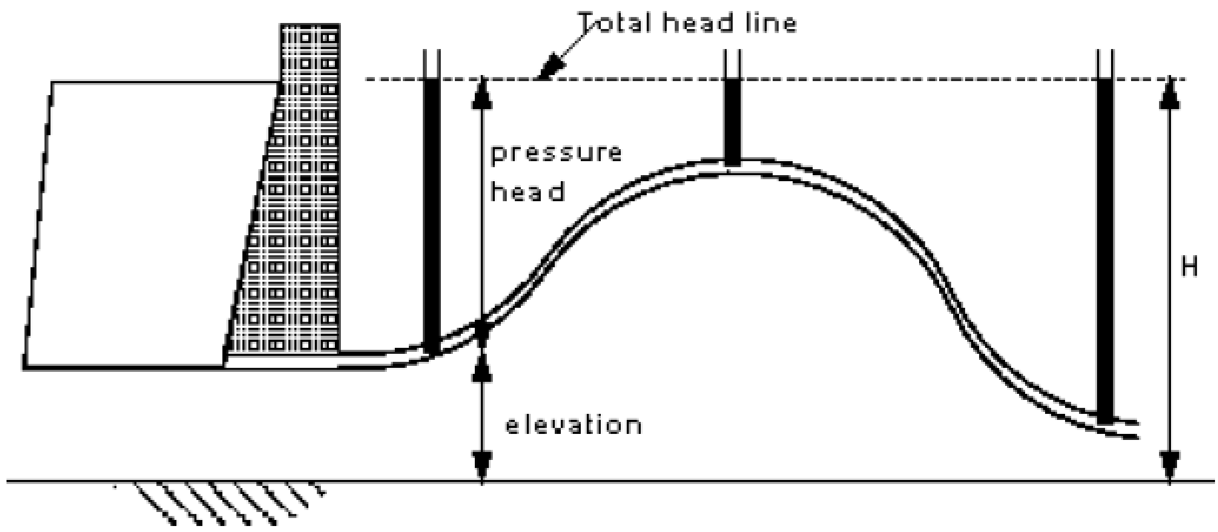
Consider the following reservoir and pipe system:



The energy equation gives us:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 = H = \text{constant}$$

Taking there to be zero velocity everywhere, we can draw this total head on the diagram:

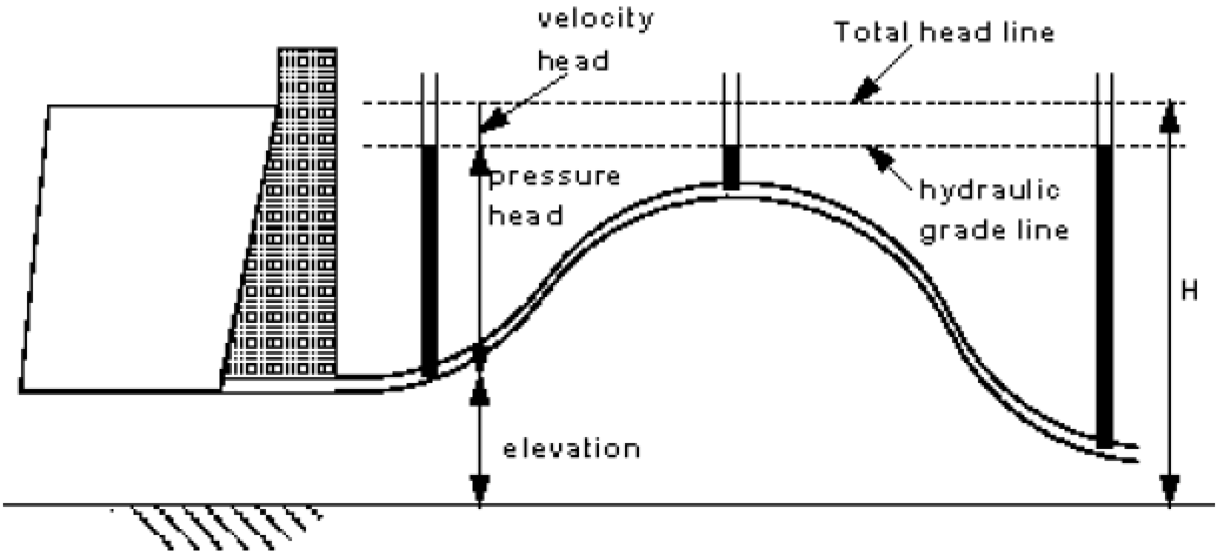


Hence at each point we have an exchange between pressure head and static head:

$$\frac{p}{\rho g} + z = H$$

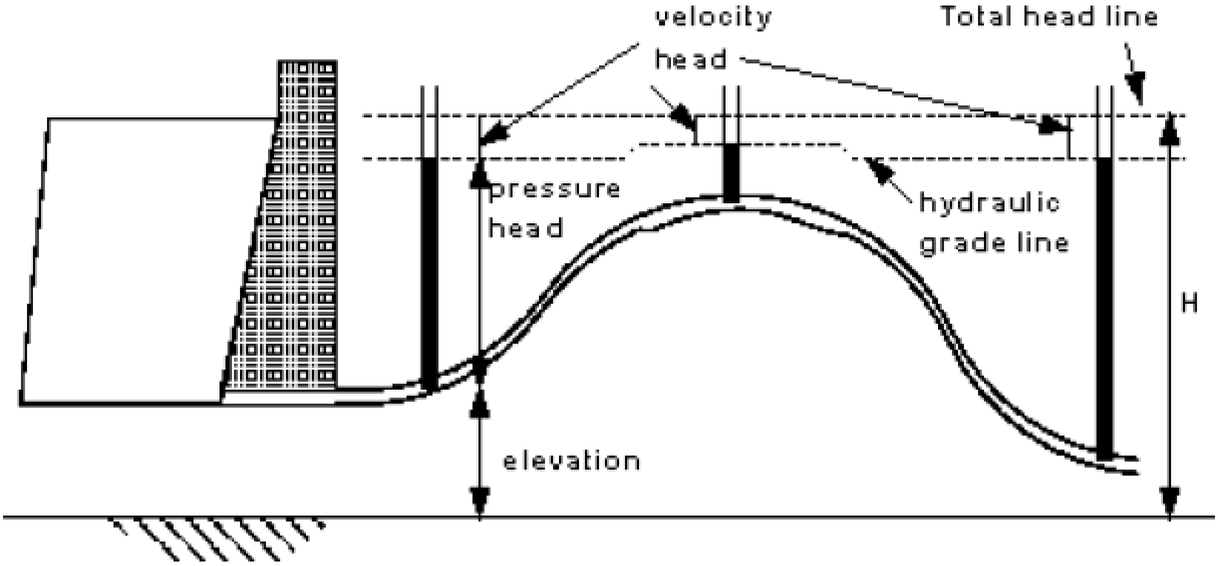
If we introduce the effect of velocity into the diagram we know that the pressure must fall by an amount $\frac{v^2}{2g}$ since we now have

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = H$$

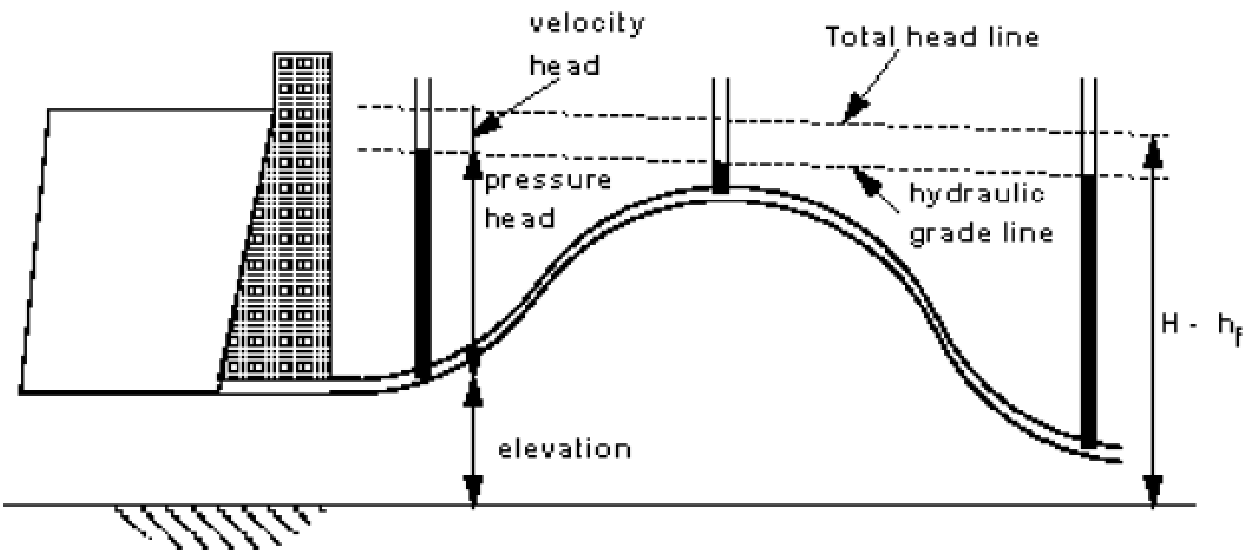


The *hydraulic grade line* is the line showing the pressure and static heads only.

If the velocity varies over the length of the pipe due to changes in diameter, say, we now have:



Note that the hydraulic grade line rises at the larger pipe section since the velocity is less in the larger pipe ($Q = Av$). If we now consider energy to be lost at every point along the length of the pipe, the total head will reduce linearly:



Thus denoting h_f as the *friction head loss*, we modify the energy equation to take account of friction losses between two points:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f$$

Problems – Energy Losses and Flow Measurement

1. Estimate the energy head lost along a short length of pipe suddenly enlarging from a diameter of 350mm to 700 mm which discharges 700 litres per second of water. If the pressure at the entrance of the flow is 10^5 N/m^2, find the pressure at the exit.

   (Ans. 0.28 m, 1.02 × 10^5 N/m^2)

2. A Venturi meter is introduced in a 300 mm diameter horizontal pipeline carrying water under a pressure of 150 kN/m^2. The throat diameter of the meter is 100 mm and the pressure at the throat is 400 mm of mercury below atmosphere. If 3% of the differential pressure is lost between the inlet and outlet throat, determine the flow rate in the pipe.

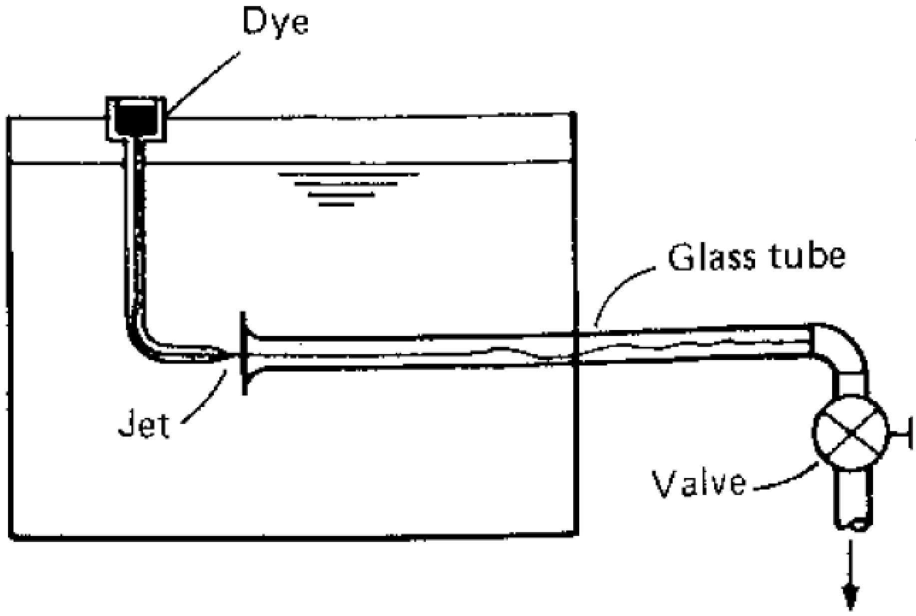
   (Ans. 157 l/s)

3. A 50 mm inlet/25 mm throat Venturi meter with a coefficient of discharge of 0.98 is to be replaced by an orifice meter having a coefficient of discharge of 0.6. If both meters are to give the same differential mercury manometer reading for a discharge of 10 l/s, determine the diameter of the orifice.

   (Ans. 31.2 mm)

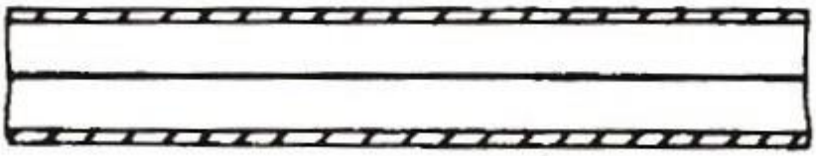
MODULE 4

The real behaviour of fluids flowing is well described by an experiment carried out by Reynolds in 1883. He set up the following apparatus:

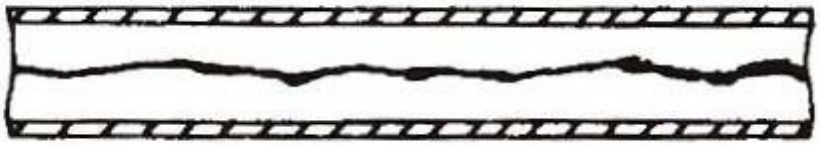


The discharge is controlled by the valve and the small 'filament' of dye (practically a streamline) indicates the behaviour of the flow. By changing the flow Reynolds noticed:

- At low flows/velocities the filament remained intact and almost straight. This type of flow is known as **laminar flow**, and the experiment looks like this:



- At higher flows the filament began to oscillate. This is called **transitional flow** and the experiment looks like:



- Lastly, for even higher flows again, the filament is found to break up completely and gets diffused over the full cross-section. This is known as **turbulent flow**:

Reynolds experimented with different fluids, pipes and velocities. Eventually he found that the following expression predicted which type of flow was found:

$$\mathrm{Re} = \frac{\rho \bar{v} l}{\mu}$$

In which Re is called the Reynolds Number; ρ is the fluid density; $\bar{v}$ is the average velocity; l is the characteristic length of the system (just the diameter for pipes), and; μ is the fluid viscosity. The Reynolds Number is a ration of forces and hence has no units.

Flows in pipes normally conform to the following:

- $\mathrm{Re} < 2000$: gives laminar flow;

- $2000 < \mathrm{Re} < 4000$: transitional flow;
- $\mathrm{Re} > 4000$: turbulent flow.

These values are only a rough guide however. Laminar flows have been found at Reynolds Numbers far beyond even 4000.

For example, if we consider a garden hose of 15 mm diameter then the limiting average velocity for laminar flow is:

$$\mathrm{Re} = \frac{\rho \bar{v} l}{\mu}$$

$$2000 = \frac{\left(10^{3}\right) \bar{v} \left(0.015\right)}{0.55 \times 10^{-3}}$$

$$\bar{v} = 0.073 \text{ m/s}$$

This is a very low flow and hence we can see that in most applications we deal with turbulent flow.

The velocity below which there is no turbulence is called the **critical velocity**.

Characteristics of Flow Types

For laminar flow:

- $Re < 2000$;
- 'low' velocity;
- Dye does not mix with water;
- Fluid particles move in straight lines;
- Simple mathematical analysis possible;
- Rare in practical water systems.

Transitional flow

- $2000 < Re < 4000$
- 'medium' velocity
- Filament oscillates and mixes slightly.

Turbulent flow

- $Re > 4000$;
- 'high' velocity;
- Dye mixes rapidly and completely;
- Particle paths completely irregular;
- Average motion is in the direction of the flow;
- Mathematical analysis very difficult - experimental measures are used;
- Most common type of flow.

Background to Pipe Flow Theory

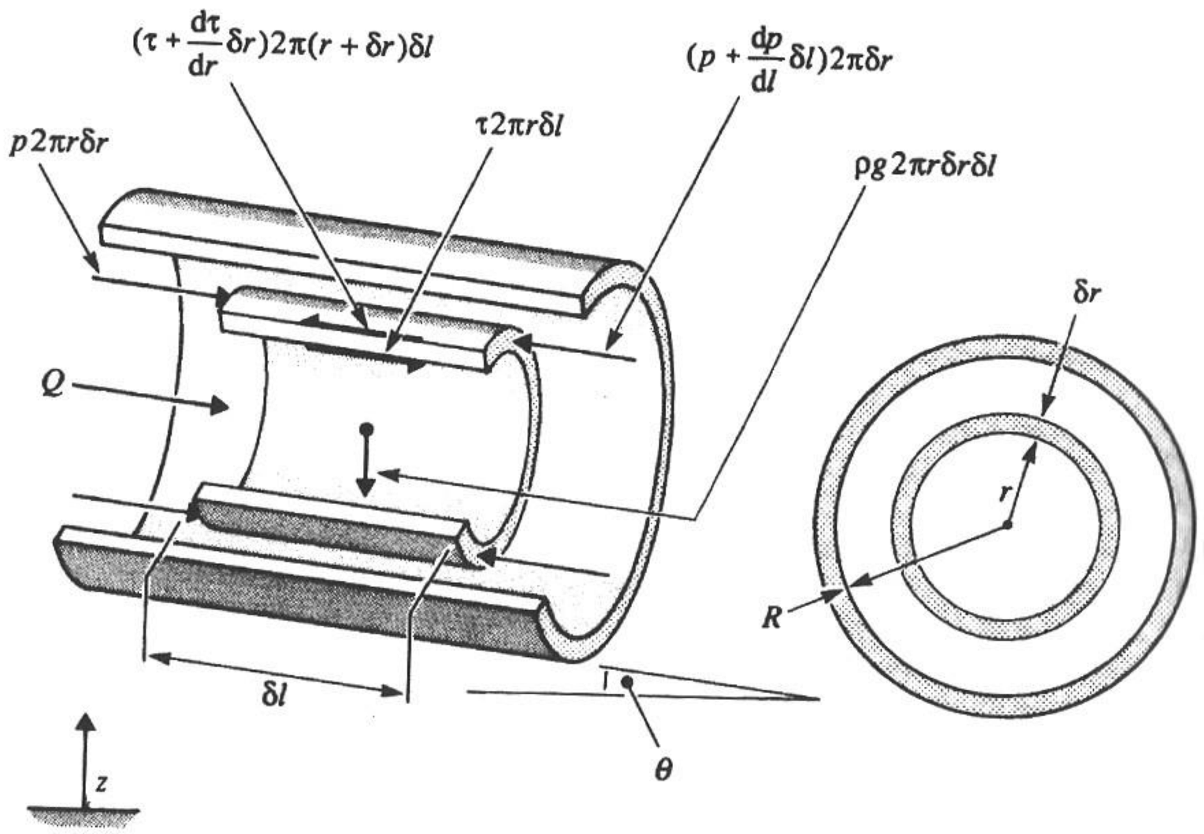
To explain the various pipe flow theories we will follow the historical development of the subject:

Date	Name	Contribution
~1840	Hagen and Poiseuille	Laminar flow equation
1850	Darcy and Weisbach	Turbulent flow equation
1883	Reynolds	Distinction between laminar and turbulent flow
1913	Blasius	Friction factor equation or smooth pipes
1914	Stanton and Pannell	Experimental values of friction factor for smooth pipes
1930	Nikuradse	Experimental values of friction factor for artificially rough pipes
1930s	Prandtl and von Karman	Equations for rough and smooth friction factors
1937	Colebrook and White	Experimental values of the friction factor for commercial pipes and the transition formula
1944	Moody	The Moody diagram for commercial pipes
1958	Ackers	Hydraulics Research Station charts and tables for the design of pipes and channels
1975	Barr	Solution of the Colebrook-White equation

Laminar Flow

Steady Uniform Flow in a Pipe: Momentum Equation

The development that follows forms the basis of the flow theories applied to laminar flows. We remember from before that at the boundary of the pipe, the fluid velocity is zero, and the maximum velocity occurs at the centre of the pipe. This is because of the effect of viscosity. Therefore, at a given radius from the centre of the pipe the velocity is the same and so we consider an elemental annulus of fluid:



In the figure we have the following:

- δr – thickness of the annulus;
- δl – length of pipe considered;
- R – radius of pipe;
- θ – Angle of pipe to the horizontal.

The forces acting on the annulus are:

- The pressure forces:
    - Pushing the fluid: $p 2\pi\, \delta r$
    - Resisting $\left(p + \frac{dp}{dl} \delta l \right) 2\pi \delta r$

- The shear forces (due to viscosity):
    - Inside the annulus: $\tau\, 2\pi r \delta l$
    - Outside the annulus $\left(\tau + \frac{d\tau}{dr} \delta r \right) 2\pi \left(r + \delta r \right) \delta l$

- The weight of the fluid (due to the angle θ):

$$\rho g\, 2\pi \delta l \delta r \sin\theta$$

The sum of the forces acting is equal to the change in momentum. However, the change in momentum is zero since the flow is steady and uniform. Thus:

$$p 2\pi \delta r - \left(p + \frac{dp}{dl} \delta l \right) 2\pi \delta r + \tau\, 2\pi r \delta l - \left(\tau + \frac{d\tau}{dr} \delta r \right) 2\pi \left(r + \delta r \right) \delta l + \rho g\, 2\pi \delta l \delta r \sin\theta = 0$$

Using $\sin\theta = -dz/dl$, and dividing by $2\pi r \delta l \delta r$ gives:

$$-\frac{dp}{dl} - \frac{d\tau}{dr} - \frac{\tau}{r} - \rho g \frac{dz}{dl} = 0$$

In which second order terms have been ignored. We introduce the term $p^* = p + \rho g z$ which is the piezometric pressure measured from the datum $z = 0$ to give:

$$-\frac{dp^*}{dl} - \left(\frac{\tau}{r} + \frac{d\tau}{dr}\right) = 0$$

Examining the term in brackets, we see:

$$\frac{d\tau}{dr} + \frac{\tau}{r} = \frac{1}{r}\left(r\frac{d\tau}{dr} + \tau\right) = \frac{1}{r}\frac{d}{dr}(\tau r)$$

Hence:

$$-\frac{dp^*}{dl} - \frac{1}{r}\frac{d}{dr}(\tau r) = 0$$

$$\frac{d}{dr}(\tau r) = -r\frac{dp^*}{dl}$$

Integrating both sides:

$$\tau r = -\frac{r^2}{2}\frac{dp^*}{dl} + C$$

But at the centreline, $r = 0$ and thus the constant of integration $C = 0$. Thus:

$$\boxed{\tau = -\frac{r}{2}\frac{dp^*}{dl}}$$

Thus the shear stress at any radius is known in terms of the piezometric pressure.

Hagen-Poiseuille Equation for Laminar Flow

We can use the knowledge of the shear stress at any distance from the centre of the pipe in conjunction with our knowledge of viscosity as follows:

$$\tau = \mu \frac{dv}{dy} = -\mu \frac{dv_r}{dr}$$

$$= -\frac{r}{2} \frac{dp^*}{dl}$$

Hence:

$$\frac{dv_r}{dr} = \frac{r}{2\mu} \frac{dp^*}{dl}$$

Integrating:

$$v_r = \frac{r^2}{4\mu} \frac{dp^*}{dl} + C$$

At the pipe boundary, $v_r = 0$ and $r = R$, Hence we can solve for the constant as:

$$C = -\frac{R^2}{4\mu} \frac{dp^*}{dl}$$

And so:

$$\boxed{v_r = -\frac{1}{4\mu} \frac{dp^*}{dl} \left(R^2 - r^2 \right)}$$

Thus the velocity distribution is parabolic (i.e. a quadratic in *r*). The total discharge can now be evaluated:

$$\delta Q = \left(2\pi r \delta r\right) v_r$$

Introducing the equation for the velocity at radius *r* and integrating gives:

$$\begin{aligned} Q &= 2\pi \int_0^R r v_r \, dr \\ &= -\frac{2\pi}{4\mu} \frac{dp^*}{dl} \int_0^R r\left(R^2 - r^2\right) dr \\ &= -\frac{\pi}{8\mu} \frac{dp^*}{dl} R^4 \end{aligned}$$

The mean velocity, $\bar{v}$ is obtained from *Q* as:

$$\begin{aligned} \bar{v} &= \frac{Q}{A} \\ &= -\frac{\pi}{8\mu} \frac{dp^*}{dl} R^4 \frac{1}{2\pi R^2} \\ &= -\frac{1}{8\mu} \frac{dp^*}{dl} R^2 \end{aligned}$$

At this point we introduce the allowance for the frictional head loss, which represents the change in pressure head occurring over the length of pipe examined, i.e.:

$$h_f = -\frac{\Delta p^*}{\rho g}$$

Therefore, introducing this and the relation for the pipe diameter $R^2 = D^2/4$, the equation for the mean velocity becomes:

$$\bar{v} = \frac{1}{8\mu}\frac{h_f}{L}\rho g\frac{D^2}{4}$$

And rearranging for the head loss that occurs, gives the Hagen-Poiseuille Equation:

$$\boxed{h_f = \frac{32\mu L\bar{v}}{\rho g D^2}}$$

Example: Laminar Flow in Pipe

Problem

Oil flows through a 25 mm diameter pipe with mean velocity of 0.3 m/s. Given that the viscosity $\mu = 4.8 \times 10^{-2}$ kg/ms and the density $\rho = 800$ kg/m^3, calculate: (a) the friction head loss and resultant pressure drop in a 45 m length of pipe, and; (b) the maximum velocity, and the velocity 5 mm from the pipe wall.

Solution

Firstly check that the laminar flow equations developed apply, that is, Re < 2000:

$$\begin{aligned}
\mathrm{Re} &= \frac{\rho D \bar{v}}{\mu} \quad \text{for pipe flow} \\
&= \frac{(800)(0.025)(0.3)}{4.8 \times 10^{-2}} \\
&= 125 \\
&< 2000 \text{ thus laminar equations apply}
\end{aligned}$$

1. To find the friction head loss, we apply the Hagen-Poiseuille Equation:

$$\begin{aligned}
h_f &= \frac{32 \mu L \bar{v}}{\rho g D^2} \\
&= \frac{32\left(4.8 \times 10^{-2}\right)(45)(0.3)}{(800)(9.81)(0.025)^2} \\
&= 4.228 \text{ m of oil}
\end{aligned}$$

The associated pressure drop is:

$$\Delta p = -\rho g h_f$$
$$= -(800)(9.81)(4.228)$$
$$= -33.18 \text{ kN/m}^2$$

The negative sign is used to enforce the idea that it is a pressure drop.

2. To find the velocities, use the equation for velocities at a radius:

$$v_r = -\frac{1}{4\mu}\frac{dp^*}{dl}\left(R^2 - r^2\right)$$

The maximum velocity occurs furthest from the pipe walls, i.e. at the centre of the pipe where $r = 0$, hence:

$$v_{\max} = -\frac{1}{4\left(4.8\times10^{-2}\right)}\frac{\left(-33.18\times10^3\right)}{45}\left(\left(\frac{0.025}{2}\right)^2 - 0\right)$$
$$= 0.6 \text{ m/s}$$

Note that the maximum velocity is twice the mean velocity. This can be confirmed for all pipes algebraically. The velocity at 5 mm from the wall is:

$$v_{\max} = -\frac{1}{4\left(4.8\times10^{-2}\right)}\frac{\left(-33.18\times10^3\right)}{45}\left(\left(\frac{0.025}{2}\right)^2 - \left(0.0075\right)^2\right)$$
$$= 0.384 \text{ m/s}$$

In which it must be remembered that at 5 mm from the wall, $r = \frac{25}{2} - 5 = 7.5 \text{ mm}$.

Turbulent Flow

Description

Since the shearing action in laminar flows is well understood, equations describing the flow were easily determined. In turbulent flows there is no simple description of the shear forces that act in the fluid. Therefore the solutions of problems involving turbulent flows usually involve experimental results.

In his work, Reynolds clarified two previous results found experimentally:

- Hagen and Poiseuille found that friction head loss is proportional to the mean velocity:

$$h_f \propto \bar{v}$$

  Reynolds found that this only applies to laminar flows, as we have seen.

- Darcy and Weisbach found that friction head loss is proportional to the mean velocity squared:

$$h_f \propto \bar{v}^2$$

  Reynolds found that this applies to turbulent flows.

Empirical Head Loss in Turbulent Flow

Starting with the momentum equation previously developed, and considering only the shear stress at the pipe wall, τ_0, we have:

$$\tau_0 = -\frac{R}{2}\frac{dp^*}{dl}$$

We also know from the Hagen-Poiseuille equation that:

$$-\frac{dp^*}{dl} = \frac{h_f \rho g}{L}$$

Hence:

$$\tau_0 = \frac{h_f}{L}\rho g \frac{R}{2}$$

Using he experimental evidence that $h_f \propto \bar{v}^2$, we introduce $h_f = K_1\bar{v}^2$:

$$\begin{aligned}\tau_0 &= \frac{K_1\bar{v}^2}{L}\rho g \frac{R}{2} \\ &= K_2\bar{v}^2\end{aligned}$$

Hence, from previous

$$K_2\bar{v}^2 = \tau_0 = \frac{h_f}{L}\rho g \frac{R}{2}$$

And rearranging for the friction head loss:

$$h_f = \frac{4K_2 L\bar{v}^2}{\rho g D}$$

If we substitute in for some of the constants $\lambda = 8K_2/\rho$ we get:

$$\boxed{h_f = \frac{\lambda L\bar{v}^2}{2gD}}$$

This is known as the **Darcy-Weisbach Equation**.

In this equation, λ is known as the pipe friction factor and is sometimes referred to as *f* in American practice. It is a dimensionless number and is used in many design charts. It was once though to be constant but is now known to change depending on the Reynolds number and the ‘roughness’ of the pipe surface.

Pipe Friction Factor

Many experiments have been performed to determine the pipe friction factor for many different arrangements of pipes and flows.

Laminar Flow

We can just equate the Hagen-Poiseuille and the Darcy-Weisbach Equations:

$$\frac{32\mu L\bar{v}}{\rho g D^2} = \frac{\lambda L \bar{v}^2}{2gD}$$

Hence, for laminar flow we have:

$$\lambda = \frac{64\mu}{\rho D\bar{v}} = \frac{64}{\mathrm{Re}}$$

Smooth Pipes – Blasius Equation

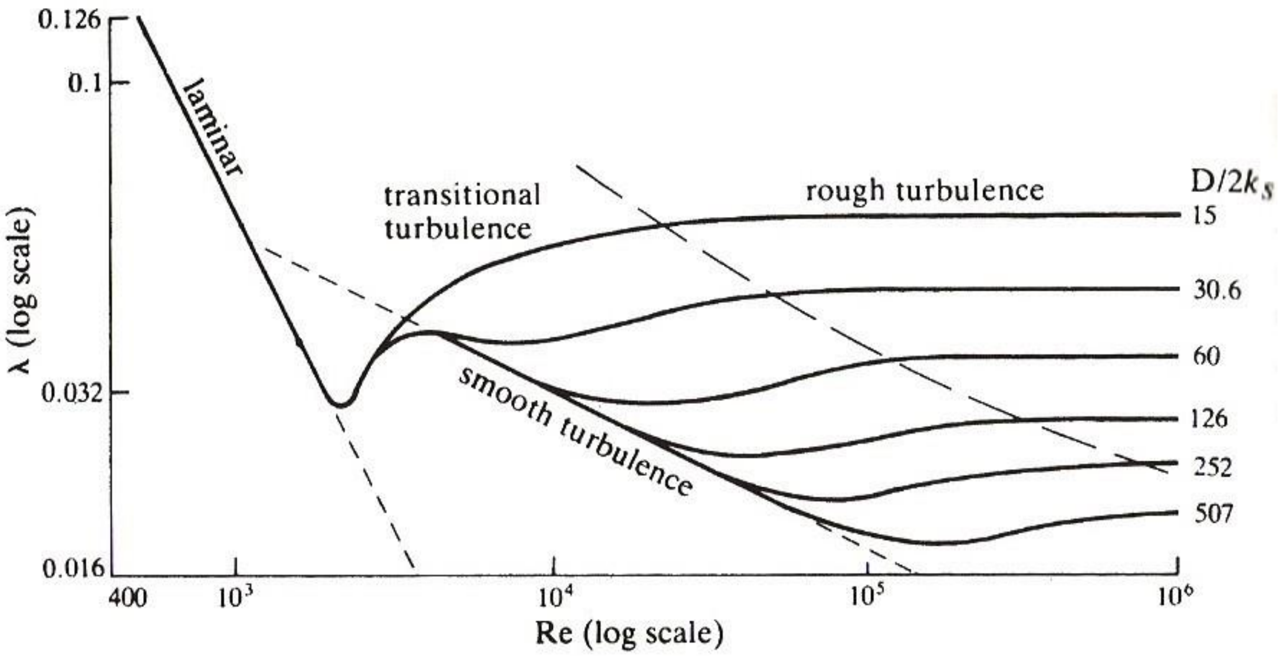
Blasius determined the following equation from experiments on ‘smooth’ pipes:

$$\lambda = \frac{0.316}{\mathrm{Re}^{0.25}}$$

Stanton and Pannell confirmed that this equation is valid for $\mathrm{Re} < 10^5$. Hence it is for ‘smooth’ pipes.

Nikuradse's Experiments

Nikuradse carried out many experiments up to $\mathrm{Re} = 3\times10^6$. In the experiments, he artificially roughened pipes by sticking uniform sand grains to smooth pipes. He defined the relative roughness (k_s/D) as the ration of the sand grain size to the pipe diameter. He plotted his results as $\log\lambda$ against $\log\mathrm{Re}$ for each k_s/D, shown below.



There are 5 regions of flow in the diagram:

1. Laminar Flow – as before;
2. Transitional flow – as before, but no clear λ;
3. Smooth turbulence – a limiting line of turbulence as Re decreases for all k_s/D;
4. Transitional turbulence – λ varies both with Re and k_s/D, most pipe flows are in this region;
5. Rough turbulence - λ is constant for a given k_s/D and is independent of Re.

The von Karman and Prandlt Laws

von Karman and Prandlt used Nikuradse's experimental results to supplement their own theoretical results which were not yet accurate. They found semi-empirical laws:

- Smooth pipes:

$$\frac{1}{\sqrt{\lambda}} = 2\log\frac{\mathrm{Re}\sqrt{\lambda}}{2.51}$$

- Rough pipes:

$$\frac{1}{\sqrt{\lambda}} = 2\log\frac{3.7}{k_s/D}$$

The von Karman and Prandlt Law for smooth pipes better fits the experimental data than the Blasius Equation.

The Colebrook-White Transition Formula

The friction factors thus far are the result of experiments on artificially roughened pipes. Commercial pipes have roughnesses that are uneven in both size and spacing. Colebrook and White did two things:

1. They carried out experiments and matched commercial pipes up to Nikuradse's results by finding an 'effective roughness' for the commercial pipes:

Pipe/Material	k_s (mm)
Brass, copper, glass, Perspex	0.003
Wrought iron	0.06
Galvanized iron	0.15
Plastic	0.03
Concrete	6.0

2. They combined the von Karman and Prandlt laws for smooth and rough pipes:

$$\boxed{\frac{1}{\sqrt{\lambda}} = -2\log\left(\frac{k_s}{3.7D} + \frac{2.51}{\mathrm{Re}\sqrt{\lambda}}\right)}$$

   This equation is known as the **Colebrook-White transition formula** and it gives results very close to experimental values for transitional behaviour when using effective roughnesses for commercial pipes.

The transition formula must be solved by trial and error and is not expressed in terms of the preferred variables of diameter, discharge and hydraulic gradient. Hence it was not used much initially.

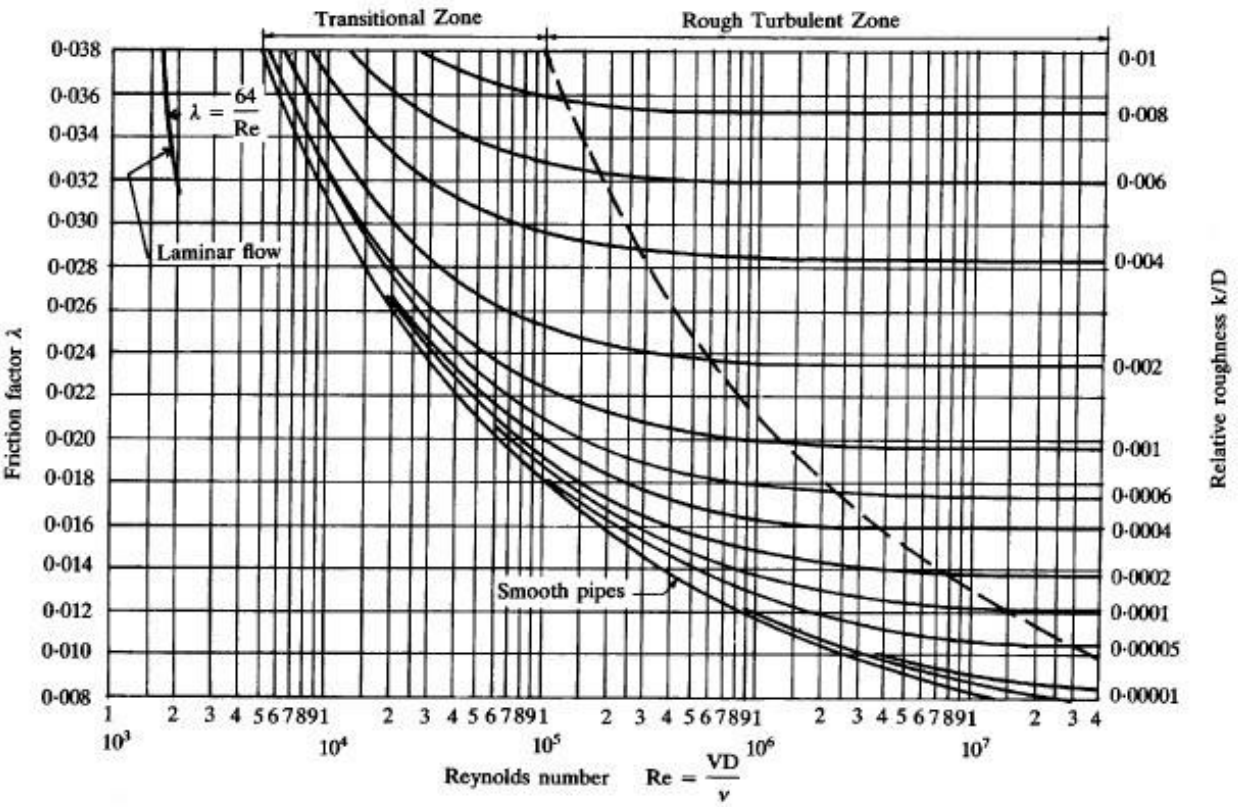
Moody

Moody recognized the problems with the Colebrok-White transition formula and did two things to remove objections to its use:

1. He presented an approximation to the Colebrook-White formula:

$$\lambda = 0.0055\left[1+\left(\frac{20\times 10^{3}k_s}{D}+\frac{10^{6}}{\mathrm{Re}}\right)^{1/3}\right]$$

2. He plotted λ against $\log \mathrm{Re}$ for commercial pipes, this is now known as the Moody diagram:



Barr

One last approximation to the Colebrook-White formula is that by Barr, who substituted the following approximation for the smooth law component:

$$\frac{5.1286}{\mathrm{Re}^{0.89}} \cong \frac{2.51}{\mathrm{Re}\sqrt{\lambda}}$$

To get:

$$\frac{1}{\sqrt{\lambda}} = -2\log\left(\frac{k_s}{3.7D} + \frac{5.1286}{\mathrm{Re}^{0.89}}\right)$$

This formula provides an accuracy of $\pm 1\%$ for $\mathrm{Re} > 10^5$.

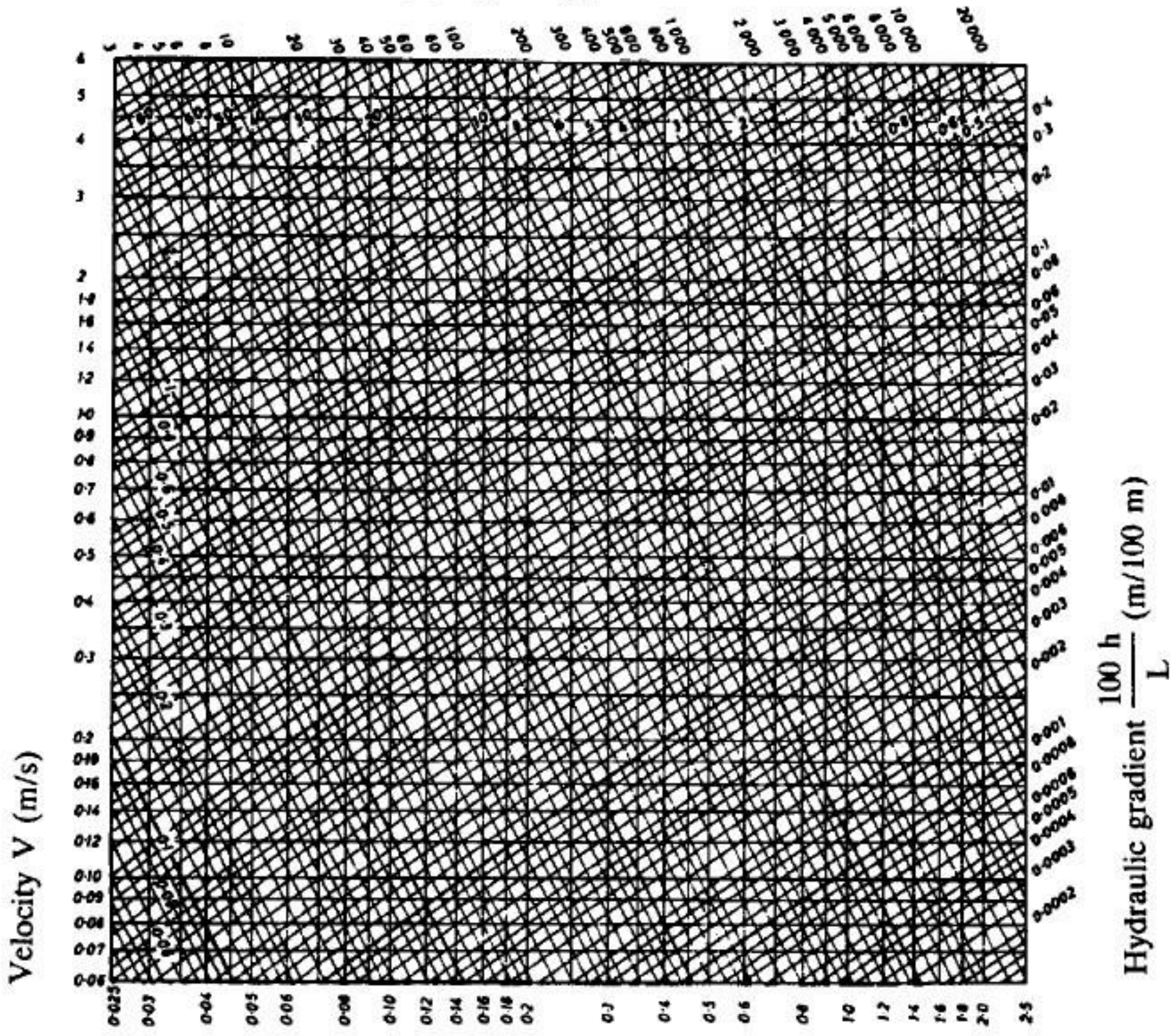
Hydraulics Research Station Charts

To derive charts suitable for design, the Colebrook-White and Darcy-Weisbach formulas were combined to give:

$$v = -2\sqrt{2gDS_f}\log\left[\frac{k_s}{3.7D} + \frac{2.51\nu}{D\sqrt{2gDS_f}}\right]$$

In which $\nu = \mu/\rho$ and is known as the kinematic viscosity and S_f is the hydraulic gradient, i.e $S_f = h_f/L$. A sample chart is:

Discharge Q (l/s) *for pipes flowing full*



Diameter D (m)

Water at 15°C

$k_s = 0{\cdot}03$ mm

Example

Problem

A plastic pipe, 10 km long and 300 mm diameter, conveys water from a reservoir (water level 850 m above datum) to a water treatment plant (inlet level 700 m above datum). Assuming the reservoir remains full, estimate the discharge using the following methods:

1. the Colebrook-White formula;
2. the Moody diagram;
3. the HRS charts.

Take the kinematic viscosity to be $1.13\times10^{-6}\ \text{m}^2/\text{s}$.

Solution

1. Using the combined Colebrook-White and Darcy-Weisbach formula:

$$\underline{v} = -2\sqrt{2gDS_f}\log\left[\frac{k_s}{3.7D} + \frac{2.51\nu}{D\sqrt{2gDS_f}}\right]$$

We have the following input variables:

1. $D = 0.3\ \text{m}$;
2. from the table for effective roughness, $k_s = 0.03\ \text{mm}$;
3. the hydraulic gradient is:

$$S_f = \frac{850-700}{10000} = 0.015$$

$$\overline{v} = -2\sqrt{2g(0.3)(0.015)}\log\left[\frac{0.03\times10^{-3}}{3.7\times0.3} + \frac{2.51\left(1.13\times10^{-6}\right)}{0.3\sqrt{2g(0.3)(0.015)}}\right]$$

$$= 2.514\ \text{m/s}$$

Hence the discharge is:

$$Q = A\bar{v} = 2.514\left(\frac{\pi\, 0.3^2}{4}\right) = 0.178 \text{ m}^3$$

2. To use the Moody chart proceed as:
    1. calculate k_s/D;
    2. assume a value for $\bar{v}$;
    3. calculate Re;
    4. estimate λ from the Moody chart;
    5. calculate h_f;
    6. compare h_f with the available head, H;
    7. if $h_f \neq H$ then repeat from step 2.

This is obviously tedious and is the reason the HRS charts were produced. The steps are:

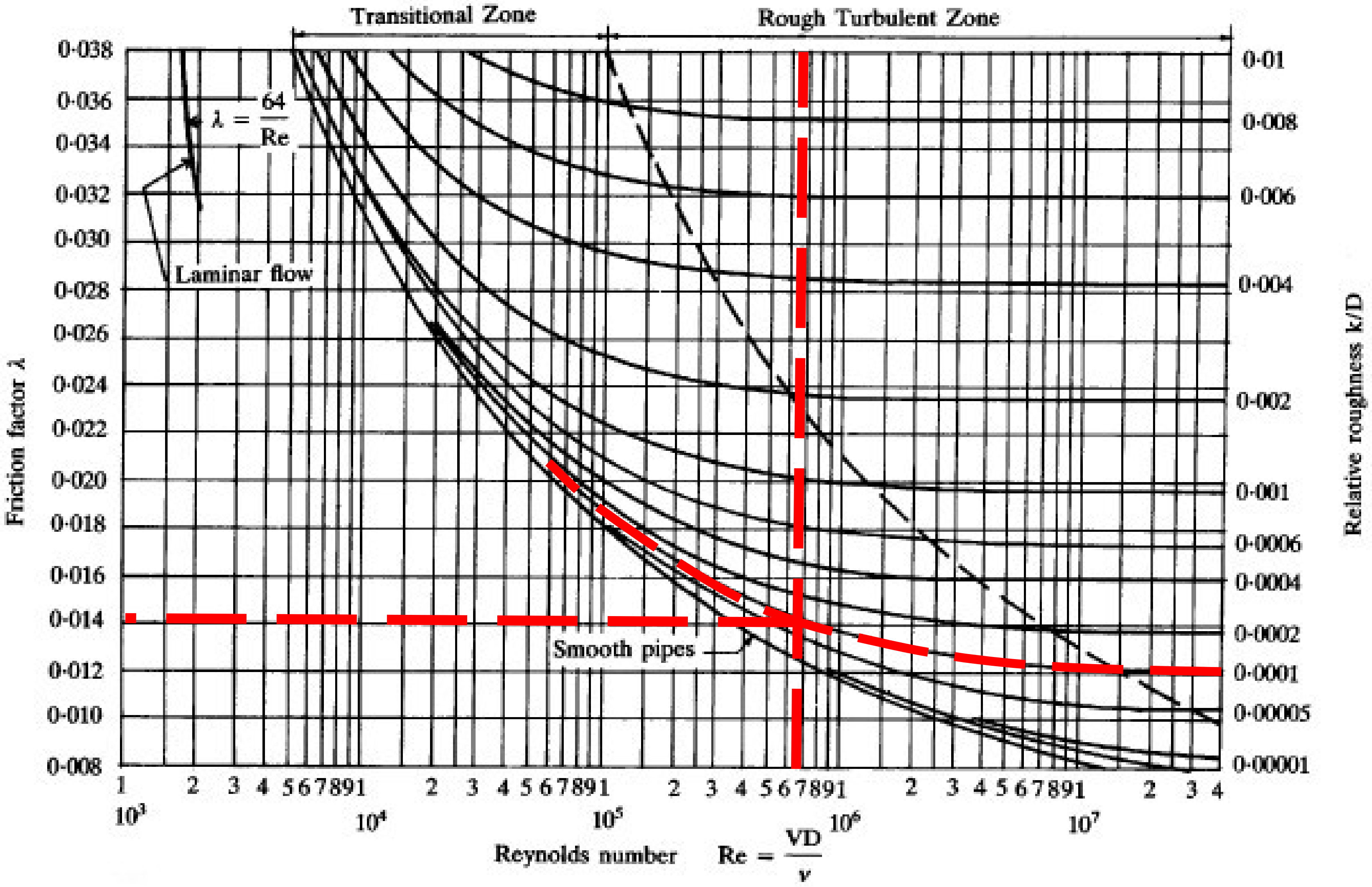
1. $k_s/D = 0.03\times10^{-3}/0.3 = 0.0001$;
2. We'll take $\bar{v}$ to be close to the known result from part 1 of the question to expedite the solution: $\bar{v} = 2.5$ m/s;
3. The Reynolds number:

$$\text{Re} = \frac{\rho\bar{v}l}{\mu} = \frac{D\bar{v}}{\nu} \text{ for a pipe}$$

$$= \frac{0.3\times 2.5}{1.13\times10^{-6}} = 0.664\times10^6$$

4. Referring to the Moody chart, we see that the flow is in the turbulent region. Follow the k_s/D curve until it intersects the Re value to get:

$$\lambda \simeq 0.014$$


Transitional Zone
Rough Turbulent Zone
$\lambda = \frac{64}{Re}$
Laminar flow
Smooth pipes
Friction factor λ
Relative roughness k/D
0·038
0·036
0·034
0·032
0·030
0·028
0·026
0·024
0·022
0·020
0·018
0·016
0·014
0·012
0·010
0·008
0·01
0·008
0·006
0·004
0·002
0·001
0·0006
0·0004
0·0002
0·0001
0·00005
0·00001
10^3
10^4
10^5
10^6
10^7
Reynolds number $Re = \frac{VD}{\nu}$

5. The Darcy-Wesibach equation then gives:

$$h_f = \frac{\lambda L v^2}{2gD}$$

$$= \frac{0.014\left(10\times 10^3\right)\left(2.5\right)^2}{2g\left(0.3\right)}$$

$$= 148.7 \text{ m}$$

6. The available head is $H = 850 - 700 = 150 \approx 148.7$ m so the result is quite close – but this is because we assumed almost the correct answer at the start.

Having confirmed the velocity using the Moody chart approach, the discharge is evaluated as before.

3. Using the HRS chart, the solution of the combined Colebrook-White and Darcy-Weisbach formula lies at the intersection of the hydraulic gradient line (sloping downwards, left to right) with the diameter (vertical) and reading off the discharge (line sloping downwards left to right):

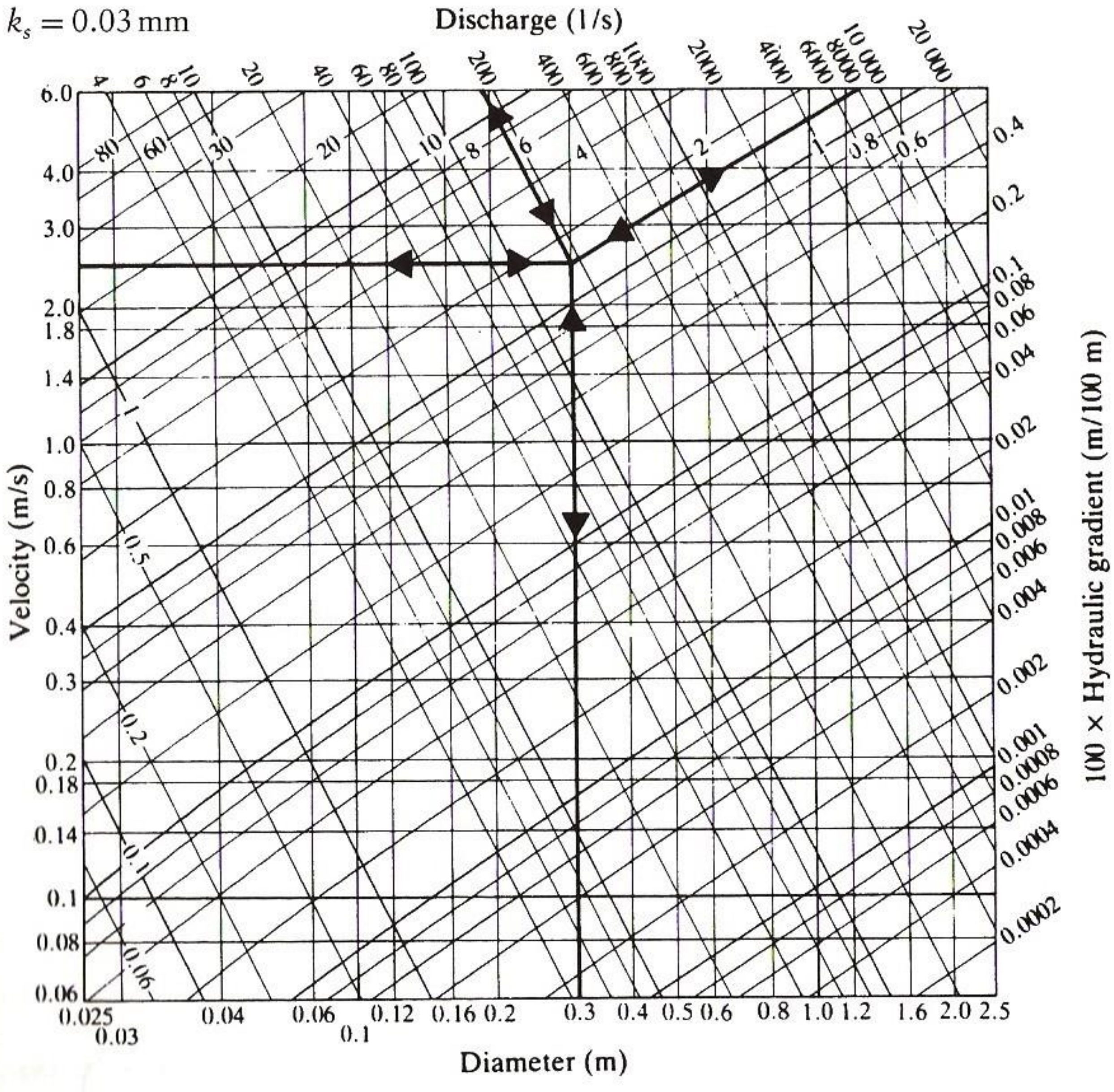
The inputs are:

- $S_f = 0.015$ and so $100S_f = 1.5$;
- $D = 300$ mm.

Hence, as can be seen from the attached, we get:

$$Q = 180 \text{ l/s}$$
$$= 0.18 \text{ m}^3\text{/s}$$

Which is very similar to the exact result calculated previously.


$k_s = 0.03$ mm
Discharge (l/s)
Velocity (m/s)
Diameter (m)
100 × Hydraulic gradient (m/100 m)

Problems – Pipe Flows

1. Determine the head loss per kilometre of a 100 mm diameter horizontal pipeline that transports oil of specific density 0.925 and viscosity 0.065 Ns/m^2 at a rate of 10 l/s. Determine also the shear stress at the pipe wall.

(Ans. 29.2 m/km, 6.62 N/m^2)

2. A discharge of 400 l/s is to be conveyed from a reservoir at 1050 m AOD to a treatment plant at 1000 m AOD. The length of the pipeline is 5 km. Estimate the required diameter of the pipe taking $k_s = 0.03\text{ mm}$.

(Ans. 450 mm)

3. The known outflow from a distribution system is 30 l/s. The pipe diameter is 150 mm, it is 500 m long and has effective roughness of 0.03 mm. Find the head loss in the pipe using:
   a. the Moody formula;
   b. the Barr formula;
   c. check these value against the Colebrook-White formula.

(Ans. 0.0182, 8.94 m)

4. A plunger of 0.08m diameter and length 0.13m has four small holes of diameter 5/1600 m drilled through in the direction of its length. The plunger is a close fit inside a cylinder, containing oil, such that no oil is assumed to pass between the plunger and the cylinder. If the plunger is subjected to a vertical downward force of 45N (including its own weight) and it is assumed that the upward flow through the four small holes is laminar, determine the speed of the fall of the plunger. The coefficient of viscosity of the oil is 0.2 kg/ms.

(Ans. 0.00064 m/s)

MODULE 5

Local Head Losses

In practice pipes have fittings such as bends, junctions, valves etc. Such features incur additional losses, termed local losses. Once again the approach to these losses is empirical, and it is found that the following is reasonably accurate:

$$h_L = k_L \frac{V^2}{2g}$$

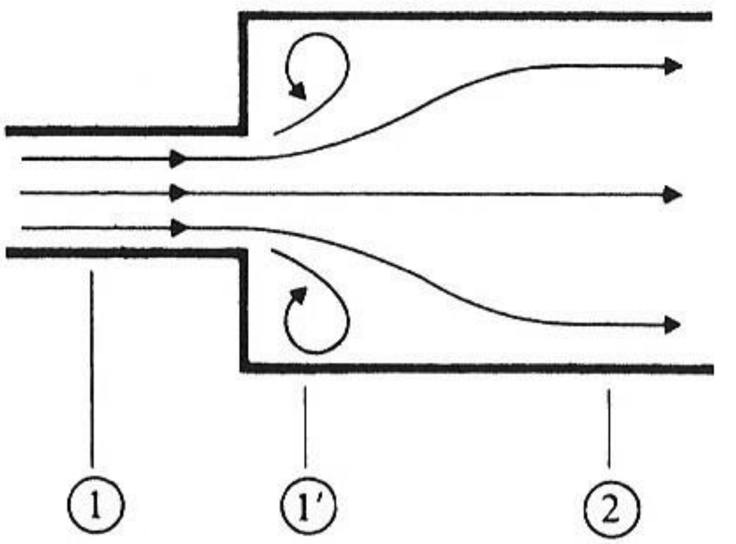
In which h_L is the local head loss and k_L is a constant for a particular fitting.

Typical values are:

Fitting	Local Head Loss Coefficient, k_L	
	Theoretical/Experimental	Design Practice
Bellmouth entrance	0.05	0.10
Bellmouth exit	0.2	0.5
90˚ bend	0.4	0.5
90˚ tees:		
- in-line flow	0.35	0.4
- branch to line	1.20	1.5
- gate valve (open)	0.12	0.25

Sudden Enlargement

Sudden enlargements (such as a pipe exiting to a tank) can be looked at theoretically:



From points 1 to 2 the velocity decreases and so the pressure increases. At 1' turbulent eddies are formed. We will assume that the pressure at 1 is the same as the pressure at 1'. Apply the momentum equation between 1 and 2:

$$p_1 A_1 - p_2 A_2 = \rho Q\left(\bar{v}_2 - \bar{v}_1\right)$$

Using continuity, $Q = A_2 \bar{v}_2$ and so:

$$\frac{p_2 - p_1}{\rho g} = \frac{\bar{v}_2}{g}\left(\bar{v}_1 - \bar{v}_2\right)$$

Now apply the energy equation from 1 to 2:

$$\frac{p_1}{\rho g} + \frac{\bar{v}_1^2}{2g} = \frac{p_2}{\rho g} + \frac{\bar{v}_2^2}{2g} + h_L$$

And so

$$h_L = \frac{\bar{v}_1^2 - \bar{v}_2^2}{2g} - \frac{p_1 - p_2}{\rho g}$$

Substituting for $\frac{p_2 - p_1}{\rho g}$ from above:

$$h_L = \frac{\bar{v}_1^2 - \bar{v}_2^2}{2g} - \frac{\bar{v}_2}{g}\left(\bar{v}_1 - \bar{v}_2\right)$$

Multiplying out and rearranging:

$$h_L = \frac{\left(\bar{v}_1 - \bar{v}_2\right)^2}{2g}$$

Using continuity again, $\bar{v}_2 = \bar{v}_1\left(A_1 / A_2\right)$ and so:

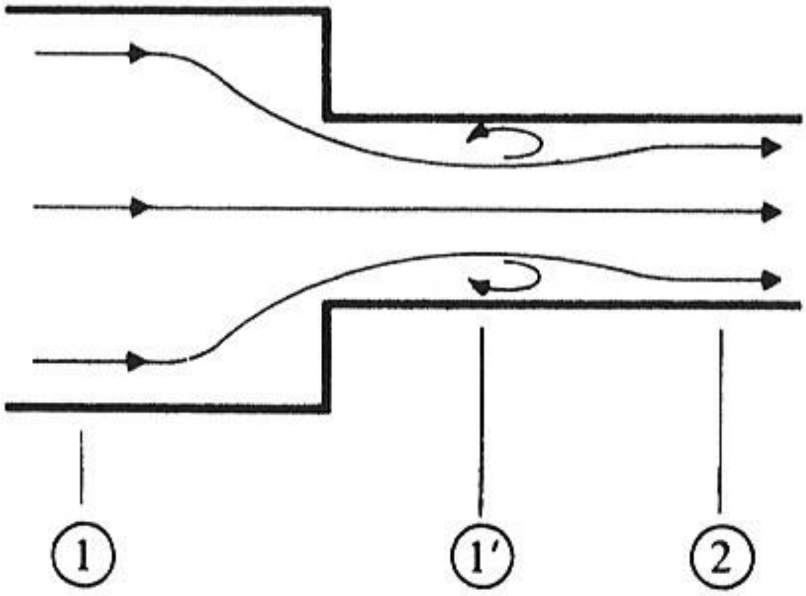
$$h_L = \frac{\left(\bar{v}_1 - \bar{v}_1 \frac{A_1}{A_2}\right)^2}{2g} = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{\bar{v}_1^2}{2g}$$

Therefore in the case of sudden contraction, the local head loss is given by:

$$k_L = \left(1 - \frac{A_1}{A_2}\right)^2$$

Sudden Contraction

We use the same approach as for sudden enlargement but need to incorporate the experimental information that the area of flow at point 1' is roughly 60% of that at point 2.



Hence:

$$A_{1'} \simeq 0.6A_2$$

$$h_L = \left(1 - \frac{0.6A_2}{A_2}\right)^2 \frac{\left(\bar{v}_2 / 0.6\right)^2}{2g}$$

$$= 0.44\frac{\bar{v}_2^2}{2g}$$

And so:

$$k_L = 0.44$$

Example – Pipe flow incorporating local head losses

Problem

For the previous example of the 10 km pipe, allow for the local head losses caused by the following items:

- 20 90˚ bends;
- 2 gate valves;
- 1 bellmouth entry;
- 1 bellmouth exit.

Solution

The available static head of 150 m is dissipated by the friction and local losses:

$$H = h_f + h_L$$

Using the table of loss coefficients, we have:

$$\begin{aligned} h_L &= \left[(20\times 0.5) + (2\times 0.25) + 0.1 + 0.5\right]\frac{\bar{v}^2}{2g} \\ &= 11.1\frac{\bar{v}^2}{2g} \end{aligned}$$

To use the Colebrook-White formula (modified by Darcy's equation) we need to iterate as follows:

1. Assume $h_f \simeq H$ (i.e. ignore the local losses for now);
2. calculate $\bar{v}$ and thus h_L;
3. calculate $h_f + h_L$ and compare to H;
4. If $H \neq h_f + h_L$ then set $h_f = H - h_L$ and repeat from 2.

From the last example, we will take $\bar{v} = 2.514 \text{ m/s}$. Thus:

$$h_L = 11.1\frac{2.514^2}{2g} = 3.58 \text{ m}$$

Adjust h_f:

$$h_f = 150 - 3.58 = 146.42 \text{ m}$$

Hence:

$$S_f = \frac{146.42}{10000} = 0.01464$$

Substitute into the Colebrook-White equation:

$$\begin{aligned} \bar{v} &= -2\sqrt{2g(0.3)(0.01464)}\log\left[\frac{0.03\times10^{-3}}{3.7\times0.3} + \frac{2.51\left(1.13\times10^{-6}\right)}{0.3\sqrt{2g(0.3)(0.01464)}}\right] \\ &= 2.386 \text{ m/s} \end{aligned}$$

Recalculate h_L:

$$h_L = 11.1\frac{2.386^2}{2g} = 3.22 \text{ m}$$

Check against H:

$$h_f + h_L = 146.42 + 3.22 = 149.64 \approx 150 \text{ m}$$

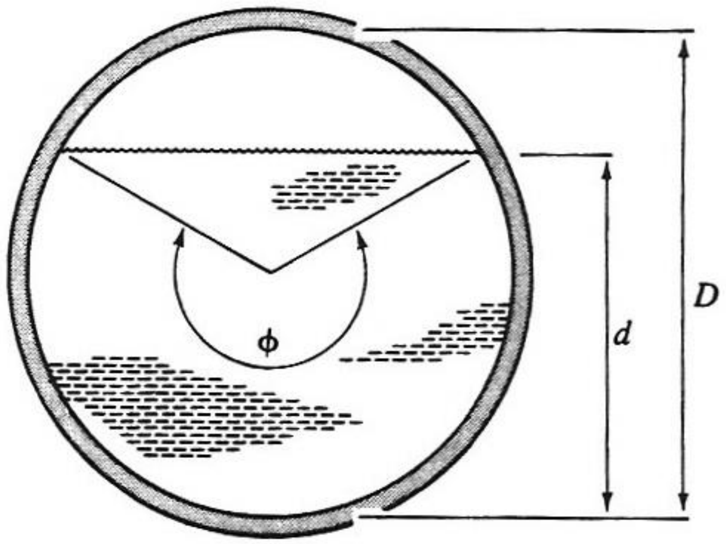
This is sufficiently accurate and gives $Q = 0.17 \text{ m}^3\text{/s}$. Note that ignoring the local losses gives $Q = 0.18 \text{ m}^3\text{/s}$, as previous.

Partially Full Pipes

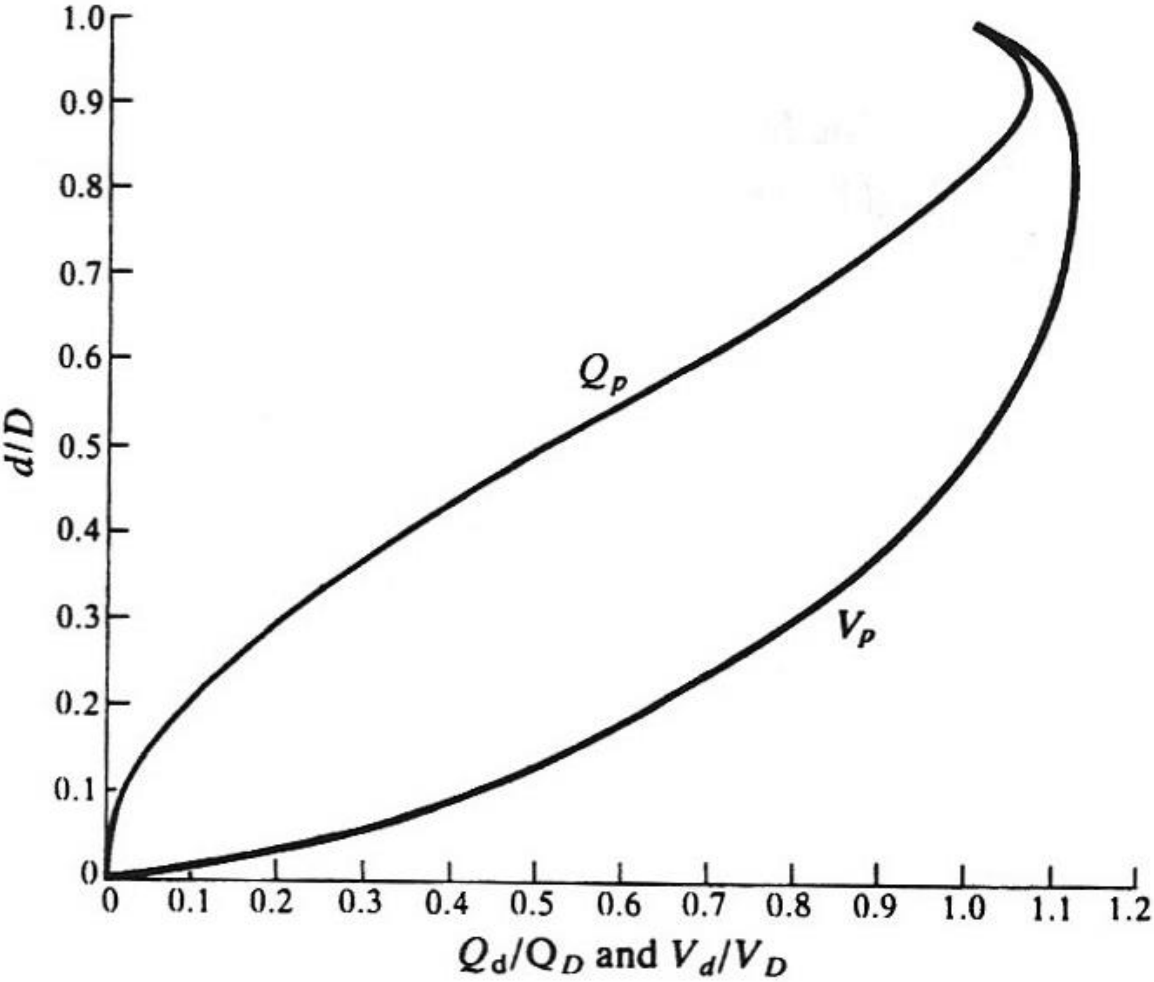
Surface water and sewage pipes are designed to flow full, but not under pressure. Water mains are designed to flow full and under pressure. When a pipe is not under pressure, the water surface will be parallel to the pipe invert (the bottom of the pipe). In this case the hydraulic gradient will equal the pipe gradient, S_0:

$$S_0 = \frac{h_f}{L}$$

In these non-pressurized pipes, they often do not run full and so an estimate of the velocity and discharge is required for the partially full case. This enables checking of the **self-cleansing velocity** (that required to keep suspended solids in motion to avoid blocking the pipe).



Depending on the proportional depth of flow, the velocity and discharge will vary as shown in the following chart:



This chart uses the subscripts: *p* for proportion; *d* for partially full, and; *D* for full.

Note that it is possible to have a higher velocity and flow when the pipe is not full due to reduced friction, but this is usually ignored in design.

Example

Problem

A sewerage pipe is to be laid at a gradient of 1 in 300. The design maximum discharge is 75 l/s and the design minimum flow is 10 l/s. Determine the required pipe diameter to carry the maximum discharge and maintain a self-cleansing velocity of 0.75 m/s at the minimum discharge.

Solution

(Note: a sewerage pipe will normally be concrete but we'll assume it's plastic here so we can use the chart for $k_s = 0.03 \text{ mm}$)

$$Q = 75 \text{ l/s}$$
$$\frac{100h_f}{L} = \frac{100}{300} = 0.333$$

Using the HRS chart for $k_s = 0.03 \text{ mm}$, we get:

$$D = 300 \text{ mm}$$
$$\bar{v} = 1.06 \text{ m/s}$$

Check the velocity for the minimum flow of 10 l/s:

$$Q_p = \frac{10}{75} = 0.133$$

Hence from the proportional flow and discharge graph:

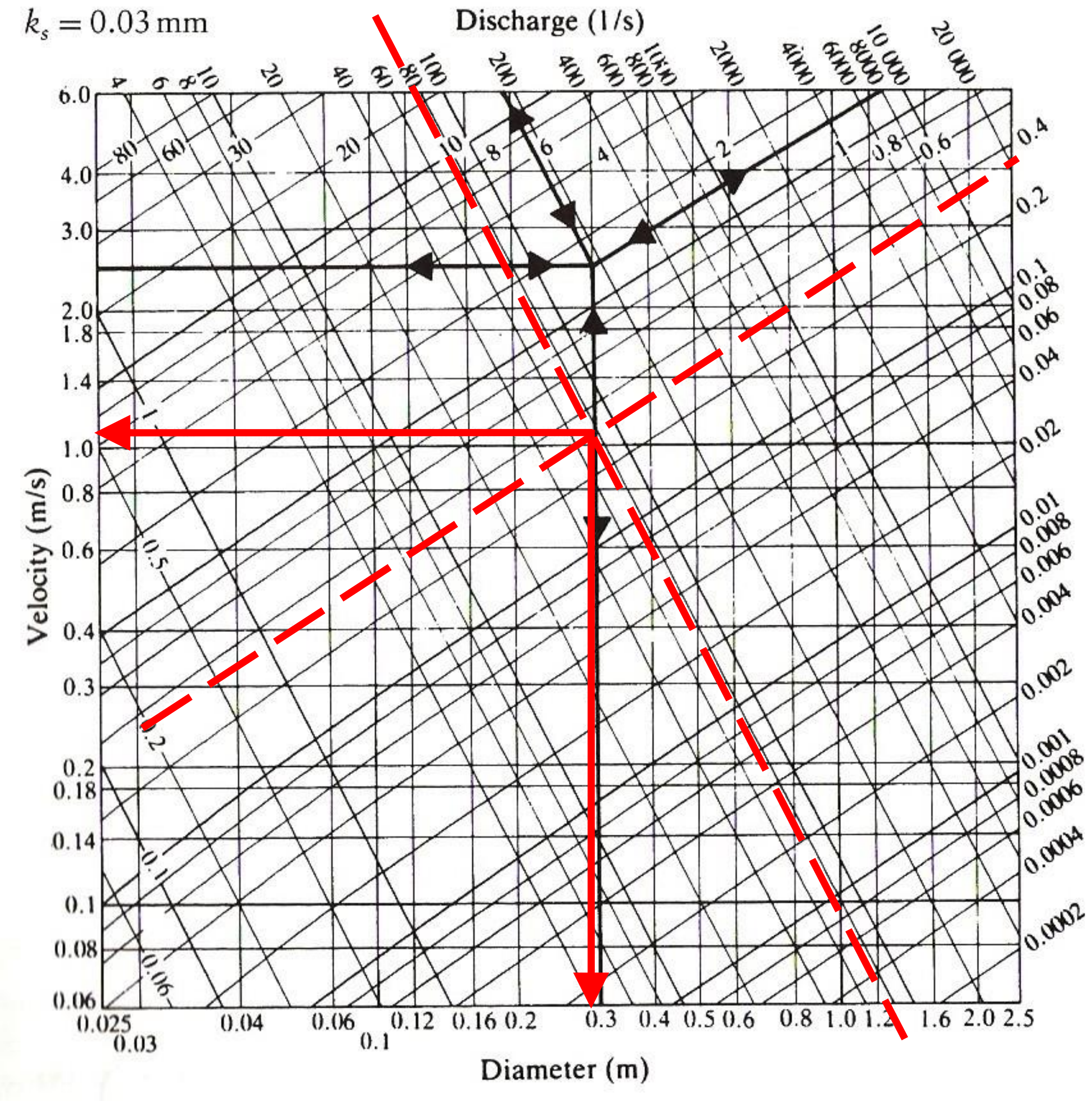
$$\frac{d}{D} = 0.25 \text{ and } v_p = 0.72$$

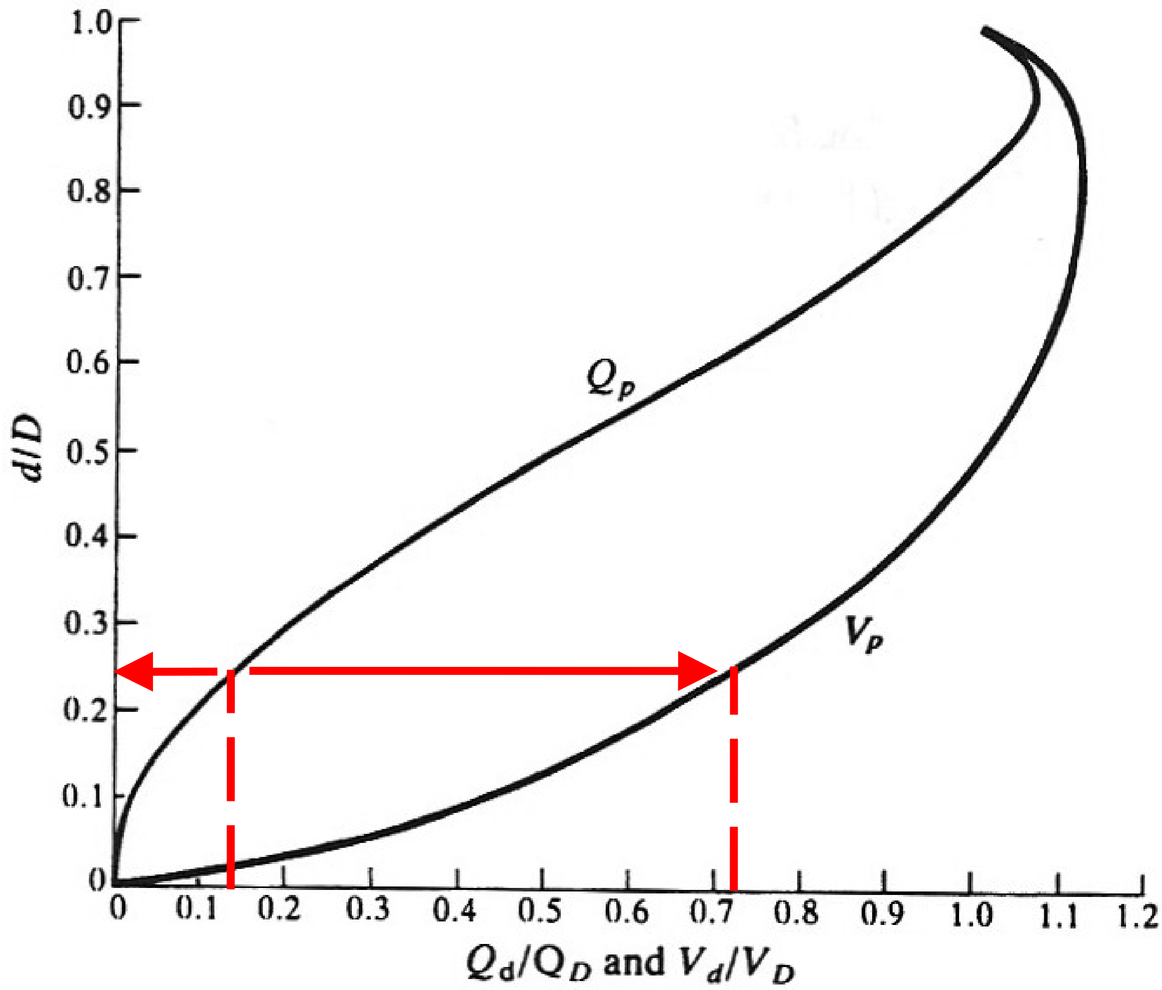
Thus:

$$\overline{v}_d = 0.72 \times 1.06 = 0.76 \text{ m/s}$$

This is greater then the minimum cleaning velocity required of 0.75 m/s and hence the 300 mm pipe is satisfactory.

The lookups are:




1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
d/D
Q_p
V_p
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
Q_d/Q_D and V_d/V_D

Problems – Pipe Design

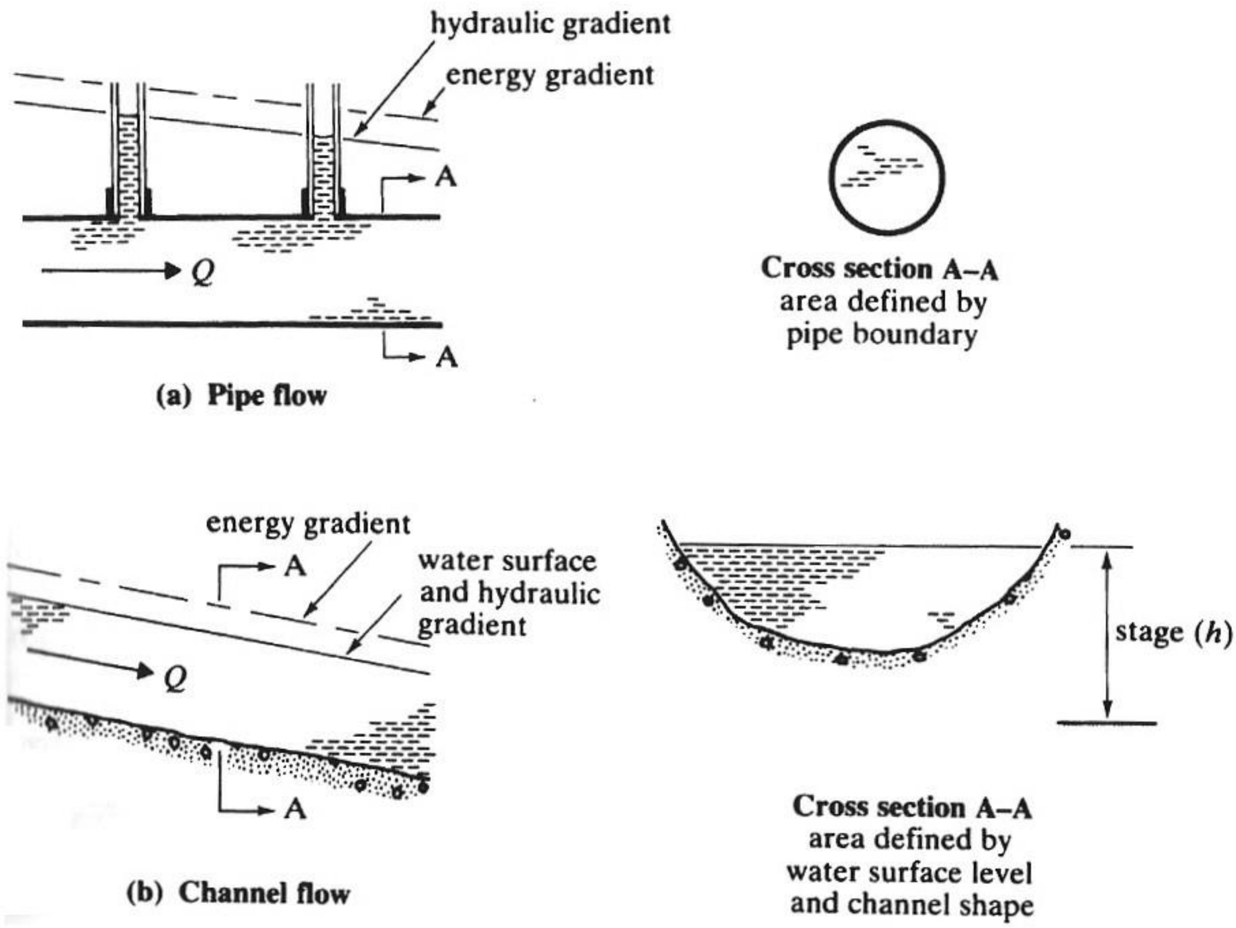
1. A uniform pipeline, 5000 m long, 200 mm in diameter and of effective roughness 0.03 mm, conveys water between two reservoirs whose surfaces are kept at a constant 50 m difference in elevation. There is an entry loss of 0.5 times the velocity head and a valve produces a loss of 10 times the velocity head. Determine the steady-state discharge using the HRS charts and confirm using the Colebrook-White transitional formula.

(Ans. 1.54 m/s, 48.4 l/s)

Hydrodynamics: Flow in Open Channels

Description

The main difference between what we have studied so far and open channels is the existence of the free surface. It has great effect as can be seen from the following comparison:



In general, the analysis of channel flow is more difficult than that of pipe flow as there are many more variables. Some approximate analyses are possible.

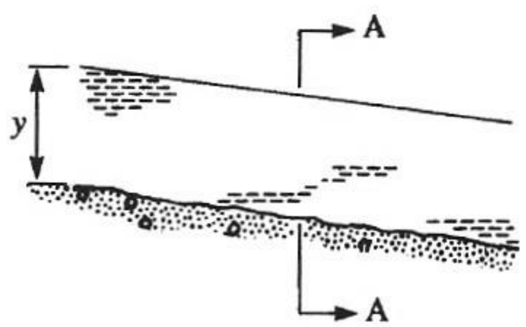
Natural channels (mainly rivers) are the most variable whilst man-made channels are more regular and thus hydrological theories are more accurate.

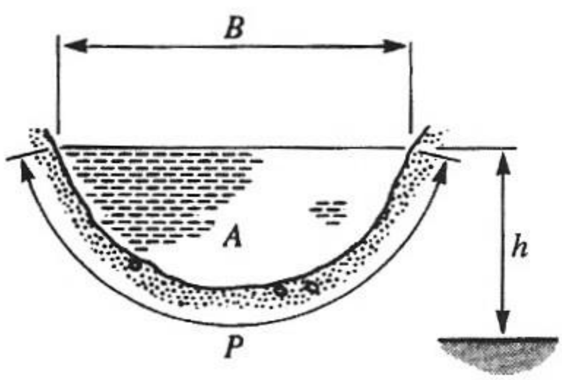
Properties

Properties used are:

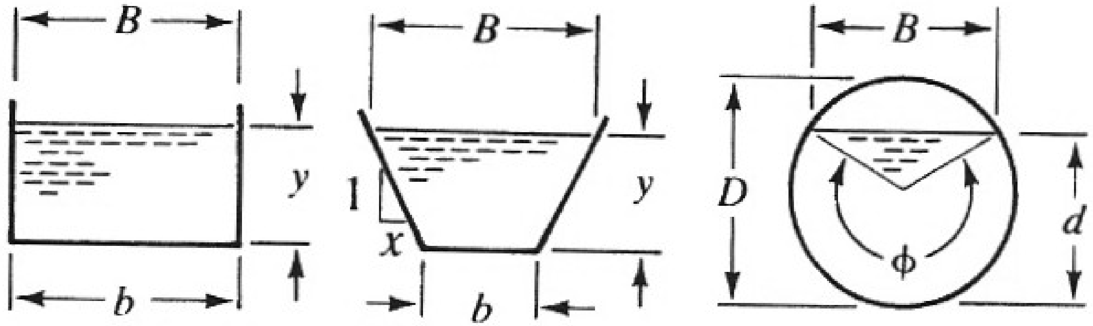
- **Depth** (y): the vertical distance from the lowest point of the channel to the free surface;
- **Stage** (h): the vertical distance from an arbitrary datum to the free surface;
- **Area** (A): the cross sectional area of flow normal to the flow direction;
- **Wetted perimeter** (P): the length of the wetted surface measured normal to the flow;
- **Surface width** (B): the width fo the channel at the free surface;
- **Hydraulic radius** (R): the ration of area to wetted perimeter (A/P);
- **Hydraulic mean depth**: the ratio of area to surface width (A/B).

The properties on a general channel are thus:





For various shapes these properties are:



	Rectangle	Trapezoid	Circle
area, A	by	$(b+xy)y$	$\frac{1}{8}(\phi - \sin\phi)D^2$
wetted perimeter, P	$b+2y$	$b+2y\sqrt{1+x^2}$	$\frac{1}{2}\phi D$
top width, B	b	$b+2xy$	$\left(\sin\frac{\phi}{2}\right)D$
hydraulic radius, R	$\frac{by}{b+2y}$	$\frac{(b+xy)y}{b+2y\sqrt{1+x^2}}$	$\frac{1}{4}\left(1-\frac{\sin\phi}{\phi}\right)D$
hydraulic mean depth, D_m	y	$\frac{(b+xy)y}{b+2xy}$	$\frac{1}{8}\left(\frac{\phi-\sin\phi}{\sin(1/2\phi)}\right)D$

Basics of Channel Flow

Laminar and Turbulent Flow

For a pipe we saw that the Reynolds Number indicates the type of flow:

$$\mathrm{Re} = \frac{\rho D\bar{v}}{\mu}$$

For laminar flow, $\mathrm{Re} < 2000$ and for turbulent flow, $\mathrm{Re} > 4000$. These results can be applied to channels using the equivalent property of the hydraulic radius:

$$\mathrm{Re}_{\mathrm{Channel}} = \frac{\rho R\bar{v}}{\mu}$$

For a pipe flowing full, $R = D/4$, hence:

$$\mathrm{Re}_{\mathrm{Channel}} = \mathrm{Re}_{\mathrm{Pipe}}/4$$

Hence:

- Laminar channel flow: $\mathrm{Re}_{\mathrm{Channel}} < 500$
- Turbulent channel flow $\mathrm{Re}_{\mathrm{Channel}} > 1000$

Moody Diagrams for Channel Flow

Using the Darcy-Weisbach equation:

$$h_f = \frac{\lambda L \bar{v}^2}{2gD}$$

And substituting for channel properties: $R = D/4$ and $h_f / L = S_0$ where S_0 is the bed slope of the channel, we have:

$$S_0 = \frac{\lambda \bar{v}^2}{8gR}$$

Hence, for a channel

$$\lambda = \frac{8gRS_0}{\bar{v}^2}$$

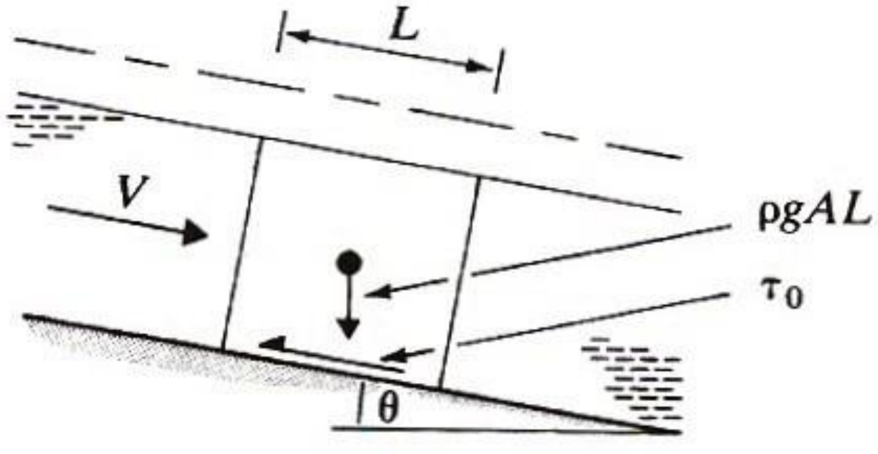
The $\lambda - \mathrm{Re}$ relationship for pipes is given by the Colebrook-White equation and so substituting $R = D/4$ and combining with Darcy's equation for channels gives:

$$\bar{v} = -2\sqrt{8gRS_0}\log\left(\frac{k_s}{14.8R} + \frac{0.6275\nu}{R\sqrt{8gRS_0}}\right)$$

A diagram, similar to that for pipes, can be drawn based on this equation to give channel velocities. This is not as straightforward though, since R varies along the length of a channel and the frictional resistance is far from uniform.

Friction Formula for Channels

For uniform flow, the gravity forces exactly balance those of the friction forces at the boundary, as shown in the diagram:



The gravity force in the direction of the flow is $\rho gAL\sin\theta$ and the shear force in the direction of the flow is $\tau_0 PL$, where τ_0 is the mean boundary shear stress. Hence:

$$\tau_0 PL = \rho gAL\sin\theta$$

Considering small slopes, $\sin\theta \approx \tan\theta \approx S_0$, and so:

$$\tau_0 = \frac{\rho gAS_0}{P} = \rho gRS_0$$

To estimate τ_0 further, we again take it that for turbulent flow:

$$\tau_0 \propto \bar{v}^2 \qquad \text{or} \qquad \tau_0 = K\bar{v}^2$$

Hence we have:

$$\bar{v} = \sqrt{\frac{\rho g}{K} R S_0}$$

Or taking out the constants gives the **Chézy Eqaution**:

$$\boxed{\bar{v} = C\sqrt{R S_0}}$$

In which C is known as the Chézy coefficient which is not entirely constant as it depends on the Reynolds Number and the boundary roughness.

From the Darcy equation for a channel we see:

$$C = \sqrt{\frac{8g}{\lambda}}$$

An Irish engineer, Robert Manning, presented a formula to give C, known as **Manning's Equation**:

$$\boxed{C = \frac{R^{1/6}}{n}}$$

In which n is a constant known as Manning's n. Using Manning;s Equation in the Chézy Equation gives:

$$\bar{v} = \frac{R^{2/3}\sqrt{S_0}}{n}$$

And the associated discharge is:

$$Q = \frac{1}{n} \frac{A^{5/3}}{P^{2/3}} \sqrt{S_0}$$

Manning's Equation is known to be both simple and reasonably accurate and is often used.

Evaluating Manning's *n*

This is essentially a roughness coefficient which determines the frictional resistance of the channel. Typical values for *n* are:

Channel type	Surface material and alignment	
river	earth, straight	0.02–0.025
	earth, meandering	0.03–0.05
	gravel (75–150 mm), straight	0.03–0.04
	gravel (75–150 mm), winding or braided	0.04–0.08
unlined canals	earth, straight	0.018–0.025
	rock, straight	0.025–0.045
lined canals	concrete	0.012–0.017
models	mortar	0.011–0.013
	Perspex	0.009

Example –Trapezoidal Channel

Problem

A concrete lined has base width of 5 m and the sides have slopes of 1:2. Manning's *n* is 0.015 and the bed slope is 1:1000:

1. Determine the discharge, mean velocity and the Reynolds Number when the depth of flow is 2 m;
2. Determine the depth of flow when the discharge is 30 cumecs.

Solution

The channel properties are:

$$A = (5 + 2y)\,y \qquad P = 5 + 2y\sqrt{1 + 2^2}$$

$$R = \frac{(5+4)2}{5 + \left(2 \times 2\sqrt{5}\right)} = 1.29 \text{ m}$$

1. Using the equation for discharge and for $y = 2$ m, we have:

$$\begin{aligned} Q &= \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_0} \\ &= \frac{1}{0.015}\frac{\left[(5+4)2\right]^{5/3}}{\left[5 + \left(2 \times 2\sqrt{5}\right)\right]^{2/3}}\sqrt{0.001} \\ &= 45 \text{ m}^3\text{/s} \end{aligned}$$

For the mean velocity, using the continuity equation:

$$v = \frac{Q}{A} = \frac{45}{(5+4)2} = 2.5 \text{ m/s}$$

And for the Reynolds number we have:

$$\mathrm{Re} = \frac{\rho R v}{\mu} = \frac{10^3 \times 1.29 \times 2.5}{1.14 \times 10^{-3}} = 2.83 \times 10^6$$

2. We have the following relationship between flow and depth:

$$Q = \frac{1}{0.015} \frac{\left[(5+2y)y\right]^{5/3}}{\left[5+2\sqrt{5}y\right]^{2/3}} \sqrt{0.001} = 2.108 \frac{\left[(5+2y)y\right]^{5/3}}{\left[5+2\sqrt{5}y\right]^{2/3}}$$

This is a difficult equation to solve and a trial an error solution is best. Since $y = 2$ m gives us 45 cumecs, try $y = 1.7$ m:

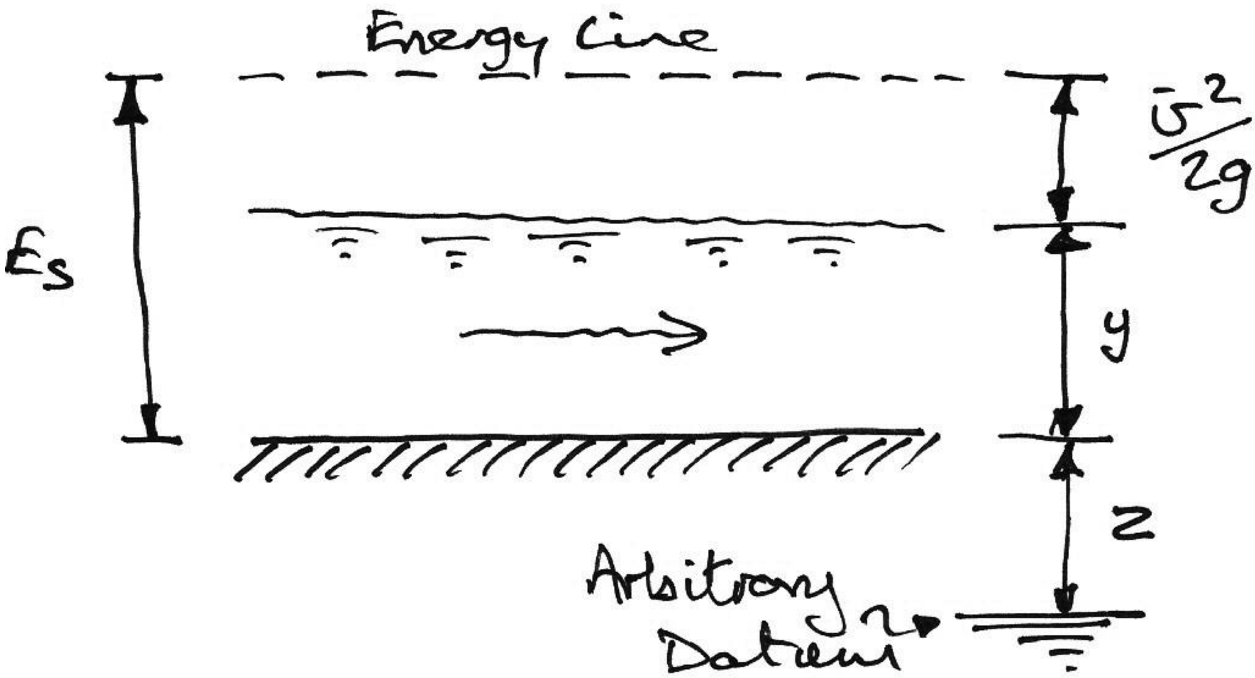
$$Q = 2.108 \frac{\left[(5+2(1.7))1.7\right]^{5/3}}{\left[5+2\sqrt{5}(1.7)\right]^{2/3}} = 32.7 \text{ m}^3/\text{s}$$

Try $y = 1.6$ m to get $Q = 29.1$ m^3/s. Using linear interpolation, the answer should be around $y = 1.63$ m for which $Q = 30.1$ m^3/s which is close enough. Hence for $Q = 30$ m^3/s, $y = 1.63$ m.

Varying Flow in Open Channels

The Energy Equation

Assuming that the channel bed is has a very small slope, the energy lines are:



Hence Bernoulli's Equation is:

$$H = y + \frac{\bar{v}^2}{2g} + z$$

To avoid the arbitrary datum, we use a quantity called the **specific energy**, E_s:

$$E_s = y + \frac{\bar{v}^2}{2g}$$

For steady flow we can write this as:

$$E_s = y + \frac{(Q/A)^2}{2g}$$

And if we consider a rectangular channel:

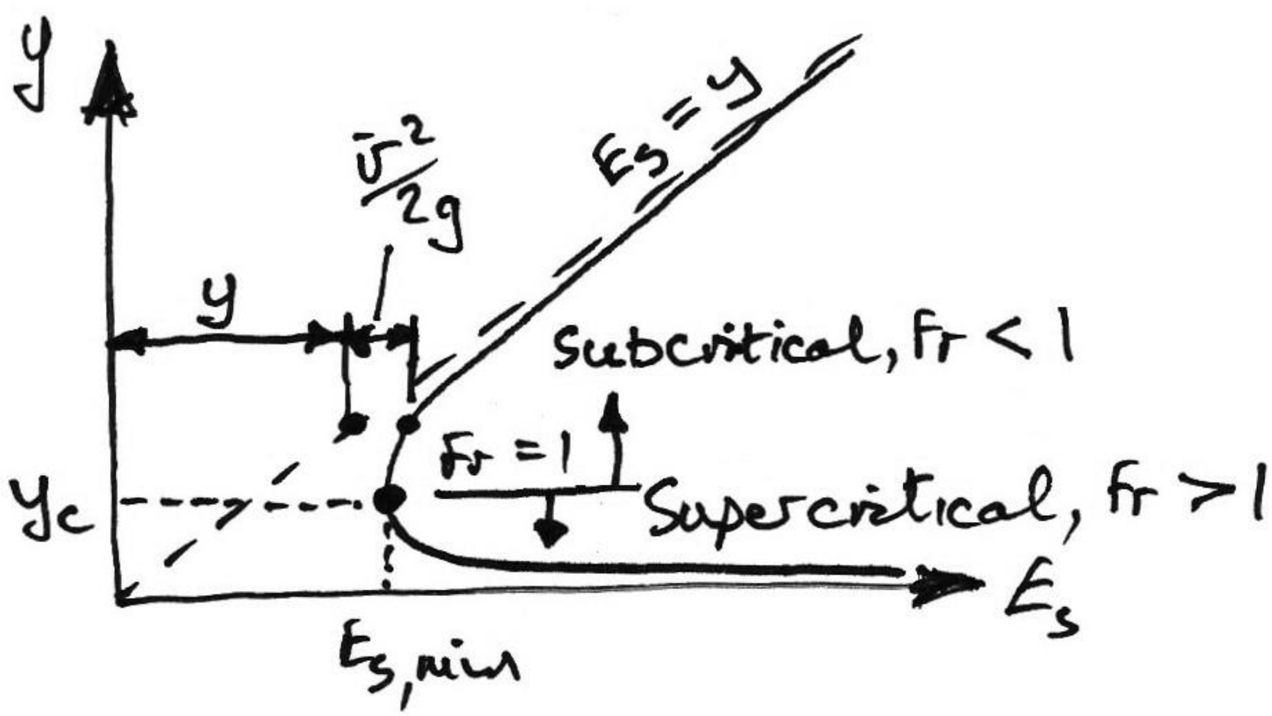
$$\frac{Q}{A} = \frac{bq}{by} = \frac{q}{y}$$

In which *q* is the mean flow per metre width of channel. Hence we have:

$$E_s = y + \frac{(q/y)^2}{2g}$$

$$(E_s - y)\,y^2 = \frac{q^2}{2g} = \text{constant}$$

This is a cubic equation in *y* for a given *q*:



Flow Characteristics

In this graph we have also identified the **Froude Number**, Fr:

$$\text{Fr} = \frac{\bar{v}}{\sqrt{gL}}$$

In which L is the characteristic length of the system. The different types of flows associated with Fr are:

- $\text{Fr} < 1$: Subcritical or tranquil flow;
- $\text{Fr} = 1$: critical flow;
- $\text{Fr} > 1$: Supercritical or rapid flow.

The Froude Number for liquids is analogous to Mach number for the speed of sound in air. In subcritical flow, a disturbance (waves) can travel up and down stream (from the point of view of a static observer). In supercritical flow, the flow is faster than the speed that waves travel at and so no disturbance travels upstream.

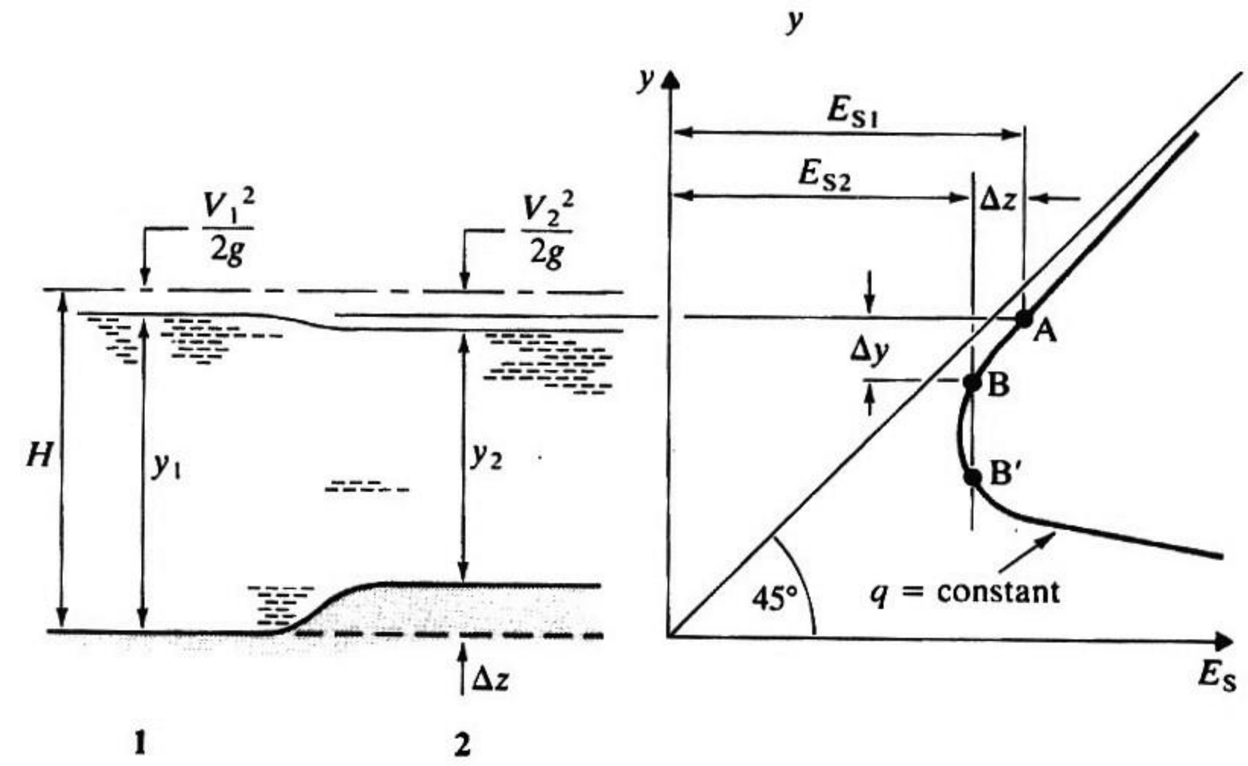
Associated with the critical flow, as shown on the graph, we have the critical depth:

$$y_c = \frac{Q^2}{gA^2}$$

A change in flow from subcritical to supercritical is termed a **hydraulic jump** and happens suddenly.

Flow Transition

Consider the situation shown where a steady uniform flow is interrupted by the presence of a hump in the streambed. The upstream depth and discharge are known; it remains to find the downstream depth at section 2.



Applying the energy equation, we have:

$$y_1 + \frac{\bar{v}_1^2}{2g} = y_2 + \frac{\bar{v}_2^2}{2g} + \Delta z$$

In addition we also have the continuity equation:

$$\bar{v}_1 y_1 = \bar{v}_2 y_2 = q$$

Combining we get:

$$y_1 + \frac{q^2}{2gy_1^2} = y_2 + \frac{q^2}{2gy_2^2} + \Delta z$$

Which gives:

$$2gy_2^3 + y_2^2\left(2g\Delta z - 2gy_1 - \frac{q^2}{y_1^2}\right) + q^2 = 0$$

Which is a cubic equation in y_2 which mathematically has three solutions, only one of which is physically admissible.

At this point refer to the specific energy curve. We see:

$$E_{s1} = E_{s2} + \Delta z$$

Also we see:

- point A on the graph represents conditions at section 1 of the channel;
- Section 2 must lie on either point B or B’ on the graph;
- All points between 1 and 2 lie on the E_s graph between A and B or B’;
- To get to B’ the river would need to jump higher than Δz (since $E_{s1} - E_{s2} > \Delta z$ between B and B’). This is physically impossible (rivers jumping?!) and so section 2 corresponds to point B.

Example – Open Channel Flow Transition

Problem

The discharge in a rectangular channel of width 5 m and maximum depth 2 m is 10 cumecs. The normal depth of flow is 1.25 m. Determine the depth of flow downstream of a section in which the river bed rises by 0.2 m over 1.0 m length.

Solution

Flow properties:

$$\bar{v} = \frac{10}{5 \times 1.25} = 1.6 \text{ m/s}$$

Using

$$E_{s1} = E_{s2} + \Delta z$$

We have:

$$E_{s1} = y_1 + \frac{\bar{v}^2}{2g} = 1.25 + \frac{1.6^2}{2g} = 1.38 \text{ m}$$

$$\mathrm{E}_{s2} = y_2 + \frac{\left[10/(5y_2)\right]^2}{2g} = y_2 + \frac{2^2}{2gy_2^2}$$

$$\Delta z = 0.2 \text{ m}$$

Hence:

$$1.38 = y_2 + \frac{2^2}{2gy_2^2} + 0.2$$
$$1.18 = y_2 + \frac{2}{gy_2^2}$$

Looking at the specific energy curve, point B must have a depth of flow less than $1.25 - 0.2 = 1.05$ m. Using a trial and error approach:

- $y_2 = 0.9$ m: Hence $E_{s2} = y_2 + \dfrac{2}{gy_2^2} = 0.9 + \dfrac{2}{g0.9^2} = 1.15 \neq 1.18$ m;
- $y_2 = 1.0$ m: Hence $E_{s2} = 1.0 + \dfrac{2}{g1.0^2} = 1.2 \neq 1.18$ m;
- $y_2 = 0.96$ m: Hence $E_{s2} = 0.96 + \dfrac{2}{g0.96^2} = 1.18$ m;

Hence the depth of flow below the hump is less than that above it due to the acceleration of the water caused by the need to maintain continuity.

Problems – Open Channel Flow

1. Measurements carried out on the uniform flow of water in a long rectangular channel 3 m wide and with a bed slope of 0.001, revealed that at a depth of flow of 0.8 m the discharge of water was 3.6 cumecs. Estimate the discharge of water using (a) the Manning equation and (b) the Darcy equation.

(Ans. 8.6 m^3/s, 8.44 m^3/s)

2. A concrete-lined trapezoidal channel has a bed width of 3.5 m, side slopes at 45° to the horizontal, a bed slope of 1 in 1000 and Manning roughness coefficient of 0.015. Calculate the depth of uniform flow when the discharge is 20 cumecs.

(Ans. 1.73 m)